Construction of half integral weight Siegel modular forms of $Sp(2, \mathbb{R})$ from automorphic forms of the compact twist $Sp(2)$

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In this paper, we construct automorphic forms of the non trivial double covering $\widetilde{Sp(2, \mathbb{R})}$ of the usual symplectic group $Sp(2, \mathbb{R})$ (matrix size four) from those of its compact twist $Sp(2) = Sp(2, \mathbb{C}) \cap U(4)$ ($U(4)$: the unitary group of size four). Our main point is that this construction preserves L functions. As well known, we have $Sp(2)/\{\pm 1\} = SO(5)$, and $(SO(5), Sp(2, \mathbb{R}))$ form a dual reductive pair defined by Howe [9], so such construction is naturally expected. Actually, one could omit $Sp(2)$ and give a formulation only on $SO(5)$, but we did not do so. For example, we formulate Hecke theory on $Sp(2)$, and not on $SO(5)$. This is because we have the following motivation. By Ihara [13] or Langlands [20], it has been conjectured that there should exist some good correspondence between automorphic forms of $Sp(n)$ and $Sp(n, \mathbb{R})$. When $n = 1$, this is Eichler's classical theorem. For $n = 2$, some examples and some good dimensional relations between these forms have been known (cf. [8], [10], [11], [12]). The only method to prove such conjecture seems to be the trace formula. It has worked well at least for dimensional relations (loc. cit.). But a more direct correspondence, if it exists, would be also very interesting. Here, instead of passing from $Sp(2)$ to $Sp(2, \mathbb{R})$, we would like to insert a "middle" term $\widetilde{Sp(2, \mathbb{R})}$, and construct the "first half" of the mapping from $Sp(2)$ to $\widetilde{Sp(2, \mathbb{R})}$. The construction from $\widetilde{Sp(2, \mathbb{R})}$ to $Sp(2, \mathbb{R})$ is left as a work in future, but we would like to point out that all the Hecke theory at finite places in this paper (e.g. comparison of local Hecke operators) remains valid also for this case, and the main obstruction for the "last half" is a lack of knowledge how to choose a correct test function at the archimedean place.

Historically, Shimura [29] has proved the correspondence between half integral weight automorphic forms of $SL(2, R)$ and integral weight forms. Our results may be regarded as a genus two version of his correspondence for the compact twist. Our technique is similar to Yoshida [32] whose origin is in Niwa [22], Shintani [30], Rallis [25], Oda [23], Kudla [17], and Howe [9]. Rallis [26] developed some local Hecke theory of the dual reductive pairs under the assumption that the double covering attached to the quadratic form is trivial. But this assumption is not satisfied in our case.

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Now, we explain our results more explicitly. Let B be a definite quaternion algebra over $\mathbb{Q}$ with discriminant d, O be a maximal order of B. Put

$$G' = \{h \in M_2(B);\ h{}^t\bar{h} = n(h)\,1_2,\ n(h) \in \mathbb{Q}^\times\},$$

where $^-$ is the main involution of B. Then, H is a $\mathbb{Q}$-form of

$$GSp(2) = \{h \in M_2(\mathbb{H});\ h{}^t\bar{h} = n(h)\,1_2,\ n(h) \in \mathbb{R}^\times\},$$

where $\mathbb{H}$ is the Hamilton quaternions. Let G'_A be the adelization of G', and G'_v be its v-component ($v \leqq \infty$). For finite primes p, put $O_p = O \otimes \mathbb{Z}_p$ and $U_p = GL_2(O_p) \cap G'_p$. Put $U = G'_\infty \prod_p U_p$. For each pair of integers (f_1, f_2) such that $f_1 \geqq f_2 \geqq 0$, denote by $(\rho_{f_1,f_2}, V_{f_1,f_2})$ the representation of G'_∞ corresponding to the Young diagram

1	$\cdots$	$\cdot$	$\cdots$	f_1
1	$\cdots$	f_2		

The space M_{f_1,f_2} of automorphic forms on G'_A with weight (f_1, f_2) belonging to U is defined by:

$$M_{f_1,f_2} = \{f: G'_A \to V_{f_1,f_2};\ f(axu) = \rho_{f_1,f_2}(u) f(x) \quad \text{for all} \quad u \in U \quad \text{and} \quad a \in G'\}$$

(cf. [5]). Here, ρ_{f_1,f_2} is regarded as the representation of G'_A by $G'_A \to G'_\infty \to GL(V_{f_1,f_2})$. On the other hand, put

$$\Gamma = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{Z});\ C \equiv 0 \bmod d' \quad \text{and} \quad \det A \equiv 1 \bmod 4\right\},$$

where d' is the least common multiple of d and 4.

We denote by $\mathrm{Sym}(f)$ the symmetric tensor representation of $\deg k'$ of $GL_2(\mathbb{C})$. For odd $k \geqq 1$, we denote by $S(\Gamma, \det^{\frac{k}{2}} \otimes \mathrm{Sym}(k'))$ the space of automorphic forms belonging to Γ with weight $\det^{\frac{k}{2}} \otimes \mathrm{Sym}(k')$. (As for the precise definition, see § 2.) On this space, Hecke operators $T_0(p^a, p^b, p^c, p^d)$ ($p \nmid d'$, $a \leqq b \leqq d \leqq c$, $a + c = b + d$) are acting (see § 3). On the other hand, denote by $T(p^a, p^b, p^c, p^d)$ the usual Hecke operators on M_{f_1,f_2}.

Main Theorem. *Assume that $f_1 + f_2$ is even. Then, there exists a $\mathbb{C}$ linear mapping*

$$\sigma: M_{f_1,f_2} \to S(\Gamma, \det^{\frac{f_1 - f_2 + 5}{2}} \otimes \mathrm{Sym}(f_2))$$

such that

$$\sigma(T(1, 1, p, p) f) = \varepsilon_p T_0(1, p, p^2, p)\, \sigma(f),$$

and

$$\sigma(T(1, p, p^2, p) f) = T_0(p, p, p^3, p^3)\, \sigma(f)$$

for all $f \in M_{f_1,f_2}$ and all primes $p \nmid d'$, where $\varepsilon_p = 1$, or i, according as $p \equiv 1 \bmod 4$, or $3 \bmod 4$.

We define in § 5 L series of elements of $S(\Gamma, \det^{\frac{k}{2}} \otimes \mathrm{Sym}(k'))$. Then, we have

Corollary. *Assumptions and notations being as above, we get $L(s, f) = L(s, \sigma(f))$ up to finitely many bad Euler factors.*

We shall treat everything adelically, because it allows us an easier treatment on Hecke theory and is more suitable for the construction by the Weil representation. In § 1, we review the p-adic double covering of symplectic groups, and extend it to the group with the square multiplicators. We also define a double covering and the Weil representation of its adelization. This is a generalization of Gelbart [3] to higher genera. In § 2, we define the mapping σ by the Weil representation, using good test functions at the infinite place as in Kashiwara and Vergne [16]. The precise definition of half-integral weight Siegel modular forms and its classical interpretation are also given there. In § 3, after a short explanation on the Hecke theory, we compare the action of Hecke operators on M_{f_1,f_2} and $S(\Gamma, \det^{\frac{f_1-f_2+5}{2}} \otimes \mathrm{Sym}(f_2))$. This part is essential. In § 5, we shall give some examples. After this work had been finished, the author had a chance to talk with Prof. Kudla, and he told me that he has obtained a correspondence between the representations of general dual reductive pairs under certain assumptions. The connection with this paper does not seem very clear at present, partly because the theory of spherical functions for the non-trivial double covering of the symplectic groups is not known (cf. [19]).

§ 1. Weil representation and double cover

In this section, we summarize some fundamental properties of the double covering of the symplectic groups with some similitudes and the Weil representation.

1. 1. Let F be any local field and put

$$Sp(n, F) = \{g \in GL_{2n}(F) : gJ{}^tg = J\},$$

where $J = \begin{pmatrix} 0 & +1_n \\ -1_n & 0 \end{pmatrix}$. The explicit 2-cocycle defining the topological double covering of $Sp(n, F)$ has been known (Weil [31], Rao [27], Perrin [24], Lion-Vergne [21]).

Now, take $Q \in M_m(F)$ such that $Q = {}^tQ$ and $\det Q \neq 0$. Then, we have an embedding as in Lions-Vergne (loc. cit.):

$$S_p(n, F) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} A \otimes 1_m, & B \otimes Q \\ C \otimes Q^{-1}, & D \otimes 1_m \end{pmatrix} \in Sp(nm, F).$$

The Weil representation R_Q of $Sp(n, F)$ attached to Q is defined by the restriction of the Weil representation of $Sp(nm, F)$ to $Sp(n, F)$ through this embedding, that is, for any C-valued L^2 function φ on $M_{n,m}(F)$, R_Q is given by the following formulae:

$$R_Q\begin{pmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{pmatrix} \varphi = \frac{\gamma(1)}{\gamma(\det a^m)} |\det a|^{\frac{m}{2}} \varphi({}^tay), \quad \text{for} \quad a \in GL_n(F),$$

$$R_Q\begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} = \chi\left(\frac{\mathrm{tr}(xyQ{}^ty)}{2}\right) \varphi(y) \quad \text{for} \quad x = {}^tx \in M_n(F),$$

$$R_Q(J) = \gamma(1)^{-nm} \int_{M_{n,m}(F)} \varphi(y')\, \chi(\mathrm{tr}(yQ{}^ty'))\, |\det Q|^{\frac{n}{2}}\, dy'.$$

Here, χ is a fixed non trivial additive character of F, and $\gamma(*)$ is a certain 8-th root of unity defined by Weil [31]. $\left(\text{Note that, in our notation, the character in Weil is } x \to \chi\left(\frac{x}{2}\right).\right)$ For any $a, b \in F$, we have

$$\frac{\gamma(ab)\,\gamma(1)}{\gamma(a)\,\gamma(b)} = (a, b)_F,$$

where $(a, b)_F$ is the Hilbert symbol on F (Weil, loc. cit.).

It is known that $R_Q(g_1 g_2) = c_Q(g_1, g_2)\, R_Q(g_1)\, R_Q(g_2)$ for some $\{\pm 1\}$ valued 2-cocycle on $Sp(n, F)$. The $c_Q(g_1, g_2)$ can be calculated explicitly for any given g_1, g_2 (cf. [21], [24], [27]):

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, F)$, put $r = \operatorname{rank} C$. Then, there exist matrices $P, Q \in GL_n(F)$ and $A_1 \in GL_{n-r}(F)$ such that $P^{-1} C Q = \begin{pmatrix} 0 & 0 \\ 0 & 1_r \end{pmatrix}$ and ${}^t P A Q = \begin{pmatrix} A_1 & * \\ 0 & * \end{pmatrix}$. Put

$$a = \begin{cases} \det C & \ldots \quad \text{if} \quad r = n, \\ \det PQ \det A_1 \ldots & \text{if} \quad r < n. \end{cases} \tag{1. 1}$$

Following [21], [24], [27],

$$t_Q(g) = \frac{\gamma(1)^{1-rm}}{(a^m (\det Q)^r)}. \tag{1. 2}$$

We denote by c_1 or t_1 the above c_Q or t_Q for the quadratic form $Q(x) = x^2$. Then, we have

$$c_Q(g_1, g_2) = c_1(g_1, g_2)^m \left(\frac{t_1(g_1)\, t_1(g_2)}{t_1(g_1 g_2)}\right)^m \frac{t_Q(g_1 g_2)}{t_Q(g_1)\, t_Q(g_2)},$$

and everything in the right hand side can be calculated for given g_1, g_2. For example, it is known by Weil that if g_1 or g_2 is an 'upper triangular' matrix, i.e. of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp(n, F)$, then

$$c_1(g_1, g_2) = \frac{t_1(g_1 g_2)}{t_1(g_1)\, t_1(g_2)}.$$

We can express c_1 more explicitly in this case. Assume that

$$\{g_1, g_2\} = \left\{\begin{pmatrix} X & Y \\ 0 & {}^t X^{-1} \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\}$$

as sets, and define a as in (1. 1) for $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then, we have

$$c_1(g_1, g_2) = (a, \det X)_F. \tag{1. 3}$$

For the sake of simplicity, we assume from now on, that $F = \mathbb{Q}_p$ or $\mathbb{R}$. To emphasize its dependence on various places v, we sometimes write c_Q, t_Q, γ, χ etc. as $c_{Q,v}$ etc. If no confusion is likely, we abbreviate Q and just write c_v.

To develop Hecke theory, we must take slightly larger groups. Put

$$G_v^+ = \{g \in M_{2n}(\mathbb{Q}_v);\ gJ{}^tg = n(g)\,J,\ n(g) \in (\mathbb{Q}_v)^2\},$$

where v is a finite or infinite place. Put $\mathscr{L} = L^2(M_{n,m}(F))$ (square integrable functions), if $F = \mathbb{R}$, and put $\mathscr{L} = \{\varphi \in L^2(M_{n,m}(F));\ \varphi(-y) = \varphi(y)$ for all $y \in M_{n,m}(F)\}$, if $F = \mathbb{Q}_p$. L is also invariant by the action of $R_Q(g)$ $(g \in Sp(n, F))$.

Proposition 1. 4. *We can extend R_Q to the representation of G_v^+ by putting*:

$$\left(R_Q\begin{pmatrix} 1_n & 0 \\ 0 & \lambda^2 1_n \end{pmatrix}\varphi\right)(y)\Big) = \varphi(\lambda^{-1}y)\,|\lambda|_v^{-m}$$

for $\varphi \in \mathscr{L}$, where λ is taken to be positive, if $F = \mathbb{R}$.

Proof. For $n = 1$, the proof has been given in Gelbart [3]. The general case is similarly proved, and we omit it here.

Remark. Actually, we can extend R_Q to G_v, where G_v is the group of all v-adic symplectic similitudes. But, it is more convenient to take the double cover of G_v^+, because, in the double cover of G_v, our important Hecke operator $T_0(1, p, p^2, p)$ vanishes identically.

From now on until the end of this paper, we fix characters χ_v as follows:

$$\chi_p(x) = \exp(-4\pi i Fr(x)), \quad \text{if} \quad F = \mathbb{Q}_p,$$

$$\chi_\infty(x) = \exp(4\pi i x), \qquad \text{if} \quad F = \mathbb{R},$$

where $Fr(x)$ is the fractional part of x. Then, $\prod_v \chi_v(x_v)$ gives a non-trivial additive character on the adeles $\mathbb{Q}_A$ which is trivial on $\mathbb{Q}$. Now, assume that $Q \in M_m(\mathbb{Z}_p)$ if $p \neq 2$, and that, for $p = 2$, Q is half-integral, that is, the diagonal components belong to $2^{-1}\mathbb{Z}_2$ and the other components to $\mathbb{Z}_2$. Put $L = M_{n,m}(\mathbb{Z}_p)$. We define the dual L' of L by:

$$L' = \{y \in M_{n,m}(\mathbb{Q}_p);\ \operatorname{tr}(yQ{}^ty') \in \mathbb{Z}_p \ \text{ for all } \ y' \in L\}.$$

Let e be the smallest nonnegative integer among those r such that $L \supset p^r L'$. Put $N_p = p^e$ if $p \neq 2$, and $N_2 = 2^{e+1}$. Put

$$K_p = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_p^+ \cap GL_2(\mathbb{Z}_p);\ C \equiv 0 \bmod N_p, \text{ and if } p = 2, \text{ also } \det A \equiv 1 \bmod 4\right\}. \tag{1.5}$$

Proposition 1. 6. *Notations and assumptions being as above, K_p splits G_p^+. More precisely, let φ be the characteristic function of L. Then, there exists a $\{\pm 1\}$ valued function s on K_p such that*

$$R_Q(k)\,\varphi = s_p(k)\,\varphi \quad \textit{for all} \quad k \in K_p.$$

Proof. It is enough to prove that $R_Q(g)\,\varphi=\varphi$, or $-\varphi$ for generators of K_p. K_p is generated by the following elements:

1) $g=\begin{pmatrix}1 & 0\\ 0 & a^2 1\end{pmatrix}$, $a\in\mathbb{Z}_p^\times$,

2) $g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$, $x={}^tx\in M_n(\mathbb{Z}_p)$,

3) $g=\begin{pmatrix}a & 0\\ 0 & {}^ta^{-1}\end{pmatrix}$, $a\in GL_n(\mathbb{Z}_p)$, and if $p=2$, also $\det a\equiv 1 \bmod 4$,

4) $g=\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$, $x={}^tx\in M_n(\mathbb{Z}_p)$, $x\equiv 0 \bmod N_p$.

As for the first three types of generators, it is easy to see that $R_Q(g)\,\varphi=\varphi$. As for the fourth, we have $g=-J\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}J$, and $R_Q(g)$ is equal to R

$$R_Q(-1)\,R_Q(J)\,R_Q\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}R_Q(J)$$

up to the sign. But, we get $R_Q(-1)\,\varphi=\varphi\gamma(1)/\gamma\big((-1)^{mn}\big)$, $R_Q(J)\,\varphi=\varphi_1\gamma(1)^{-nm}\,|\det Q|^{\frac{n}{2}}$, where φ_1 is the characteristic function of L', and

$$R_Q\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}(|\det Q|^{\frac{n}{2}}\,\varphi_1)=\gamma(1)^{-nm}\,\varphi.$$

On the other hand, $\gamma(1)/\gamma\big((-1)^{-nm}\big)\,\gamma(1)^{2nm}=\gamma(1)^{2nm}$, or $\gamma(1)^{2(1-nm)}$, according as nm is even, or odd. Because $\gamma(1)^8=1$, our Proposition is proved. q.e.d.

Now, we define an adelic double cover. Take a half-integral non-degenerate symmetric matrix $Q\in M_m(\mathbb{Q})$. Define K_p as above for each finite prime p. Prolong the above function s_p on K_p to G_p^+ by putting $s_p(g)=1$, or -1 arbitrarily for $g\subset G_p^+$, $g\notin K_p$. We fix one such prolongation and denote it also by s_p. We put $s_\infty(g)=1$ for $g\in G_\infty^+$. Put $b_v(g_1,g_2)=c_v(g_1,g_2)\,s_v(g_1)\,s_v(g_2)\,s_v(g_1g_2)$ for $v\leqq\infty$ where $c_v=$ the above 2-cocycle c_Q of $Sp(n,\mathbb{Q}_v)$. Then, $b_p=1$ on $K_p\times K_p$. Put $G=\{g\in M_{2n}(\mathbb{Q});\, gJ{}^tg=n(g)\,J,\ n(g)\in\mathbb{Q}^\times\}$, and let G_A be the adelization of G. Put $G_A^+=\{g=(g_v)\in G_A;\, n(g_v)\in(\mathbb{Q}_v^\times)^2$ for all places $v\}$. For $g_1,g_2\in G_A^+$, put

$$b(g_1,g_2)=\prod_v b_v(g_1,g_2).$$

It is clear that this is well defined. We define a double cover $\bar{G}_A^+$ of G_A^+ by this cocycle, that is, $\bar{G}_A^+=G_A^+\times\{\pm 1\}$ as a set, and the group multiplication is given by:

$$(g_1,\varepsilon_1)\,(g_2,\varepsilon_2)=\big(g_1g_2,\ \varepsilon_1\varepsilon_2 b(g_1,g_2)\big).$$

A double cover $\bar{G}_v^+$ of G_v^+ is defined in the same way by b_v. The groups K_p are subgroups of $\bar{G}_A^+$ by embedding: $K_p \ni k \longrightarrow (k, 1) \in \bar{G}_A^+$. Put

$$G^+ = \{g \in GL_{2n}(Q);\ gJ{}^tg = n(g)\,J,\ n(g) \in (Q^\times)^2\},$$

and for any $\gamma \in G^+$, put

$$s(\gamma) = \prod_v s_v(\gamma),$$

which is of course well defined. G^+ can be regarded as a subgroup of $\bar{G}_A^+$ by the mapping:

$$G^+ \ni \gamma \longrightarrow (\gamma, s(\gamma)) \in \bar{G}_A^+.$$

Proposition 1. 7. *We get* $\bar{G}_A^+ = G^+ \bar{G}_\infty^+ \prod_p K_p$.

Proof. This is obvious by virtue of the usual strong approximation theorem.

q.e.d.

Put $X = M_{n,m}(Q)$, $X_v = M_{n,m}(Q_v)$, and $X_A = M_{n,m}(Q_A)$. Denote by $S(X_A)$ the Schwartz-Bruhat functions on X_A. For a function $f = \prod_v f_v$, $f_v \in S(X_v)$, where f_p are the characteristic functions of $M_{n,m}(\mathbb{Z}_p)$ for almost all p, and

$$\bar{g} = (g, \varepsilon) \in \bar{G}_A^+ \quad (g = (g_v) \in G_A^+,\ \varepsilon = \pm 1),$$

put

$$\pi_Q(\bar{g}) f = \varepsilon \prod_v s_v(g_v)\, R_{Q,v}(g_v) f_v.$$

Such functions as above form a dense subset of $S(X_A)$, and we can extend $\pi_Q(\bar{g})$ to the action on $S(X_A)$ by continuity. We call $\pi_Q(\bar{g})$ the Weil representation of $\bar{G}_A^+$. Let V be a vector space over C. Then, we also call the representation $\pi_Q \otimes \mathrm{id}$ on $S(X_A) \otimes V$ the Weil representation.

§ 2. Automorphic forms on the double covering

In this section, we construct some automorphic forms belonging to $\bar{G}_A^+$ with $n = 2$.

2. 1. First, we define vector valued automorphic forms on $\bar{G}_A^+$. Denote the Siegel upper half space of degree n by: $\mathfrak{H}_n = \{X + iY;\ X = {}^tX,\ Y = {}^tY \in M_n(\mathbb{R}),\ Y > 0\}$. We take a function $m(g, Z)$ on $Sp(n, R) \times \mathfrak{H}_n$ as in Lions-Vergne [21] p. 174 (for our character χ_∞). Then

$$(t_\infty(g)\, m(g, Z))^2 = \det(CZ + D)^{-1}$$

for any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, where $t_\infty(g)$ is as in (1. 1) (for $Q = 1$ and $F = \mathbb{R}$). Denote by $\widetilde{Sp(n, \mathbb{R})}$ the unique double cover in $\bar{G}^+$ of $Sp(n, \mathbb{R})$. For $\bar{g} = (g, \varepsilon) \in \widetilde{Sp(n, \mathbb{R})}$, we put $J(\bar{g}, Z) = (\varepsilon m(g, Z)\, t_\infty(g))^{-1}$. Then

$$J(\bar{g}_1 \bar{g}_2, Z) = J(\bar{g}_1, g_2 Z)\, J(\bar{g}_2, Z) \text{ for } \bar{g}_i = (g_i, \varepsilon_i) \in \widetilde{Sp(n, \mathbb{R})} \quad (i = 1, 2),$$

that is, J is an automorphic factor (cf. Lions-Vergne, loc. cit.). Let (τ, V) be a finite dimensional irreducible representation of $GL_n(\mathbb{C})$.

Put $K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(n, \mathbb{R});\ A + iB \text{ is unitary} \right\}$.

Denote by $\bar{K}_\infty$ the double cover of K_∞.

Put $\Gamma = G^+ \cap \prod_p K_p$.

Definition 2. 1. Notations being the same as in § 1, assume that $Q \in M_m(\mathbb{Q})$ is half-integral positive definite and m is odd. A V valued function Φ on $\bar{G}_A^+$ is called automorphic form belonging to Γ with weight $\det^{\frac{m}{2}} \otimes \tau$, if it satisfies the following conditions:

(C 1) $\Phi(\gamma \bar{g}) = \Phi(\bar{g})$ for all $\gamma \in G^+$, $\bar{g} \in \bar{G}_A^+$,

(C 2) $\Phi(\zeta \bar{g}) = -\Phi(\bar{g})$ for $(1, -1) = \zeta \in \bar{G}_A^+$,

(C 3) $\Phi(\bar{g} k) = \Phi(\bar{g})$ for all $k \in \prod_p K_p$, $\bar{g} \in \bar{G}_A^+$,

(C 4) $\Phi(\bar{g} \bar{k}_\infty) = J(\bar{k}_\infty, i)^{-m} \tau(Ci + D)^{-1} \Phi(\bar{g})$

for all $\bar{k}_\infty = (k_\infty, \varepsilon) \in \bar{K}_\infty$, $k_\infty = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and $\bar{g} \in \bar{G}_A^+$,

(C 5) $\Phi((\lambda 1, 1)\, \bar{g}) = \Phi(\bar{g})$ for all $(\lambda 1, 1) \in \bar{G}_A^+$, $\lambda > 0$, $\bar{g} \in \bar{G}_A^+$.

The interpretation into the classical language is given as follows: For

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}),$$

put $Z = g(i) = (AZ + B)(CZ + D)^{-1}$. Put

$$f(Z) = \big(t_\infty(g)\, m(g, i)\big)^{-m} \tau(Ci + D)\, \Phi\big((g, 1)\big). \tag{2. 2}$$

Then, for any $\gamma \in \Gamma$, we have

$$f(\gamma Z) = s(\gamma)\, \big(t_\infty(\gamma)\, m(\gamma, Z)\big)^{-m} \tau(Ci + D) f(Z). \tag{2. 3}$$

Conversely, if there is a function f which satisfies (2. 3), we get an automorphic form Φ by virtue of Proposition 1. 4. The proof is standard as in Gelbert [3], and we omit it here.

2. 2. To construct automorphic forms by the Weil representation, we need some good test functions at the archimedean place. If Q is positive definite, such test functions are known by Kashiwara and Vergne [16]. We quote here the part of that theory we need. Let (λ, V_λ) be an irreducible representation of $O(5)$, where $O(5)$ is the real orthogonal group of size 5 for a positive definite form. Put

$\mathfrak{H}(\lambda) = \{ V_\lambda$ valued pluriharmonic polynomial functions $P(y)$ on $M_{2,5}(\mathbb{R})$
such that $P(yh) = \lambda(h)^{-1} P(y) \}$.

Here, $P(y)$ is called pluriharmonic, if

$$\sum_{k=1}^{5} \frac{\partial^2 P}{\partial y_{ik} \partial y_{jk}} = 0 \quad \text{for} \quad i, j = 1 \dots 5,$$

where $y = (y_{ij})$. The group $GL_2(C)$ acts on $\mathfrak{H}(\lambda)$ by: $P(y) \to P(a^{-1} y)$, $a \in GL_2(\mathbb{C})$. We denote this representation by $\tau(\lambda)$. For the sake of simplicity, we denote the Young diagram

$$\begin{array}{|l|}\hline 1 \ \dots \quad f_1 \\ \hline 1 \ \dots \ f_2 \\ \hline 1 \ \dots \\ \hline 1 \ f_r \\ \hline \end{array}$$

(or the heightest weight attached to a certain basis) by a series of integers like $(f_1, f_2, \dots, f_r)$.

Theorem 2. 4 (Kashiwara-Vergne loc. cit.). *Notations being as above, $\tau(\lambda)$ is irreducible. We have $H(\lambda) \neq 0$, if and only if λ corresponds to $(m_1, m_2; \varepsilon)$, $\varepsilon = (-1)^{m_1 + m_2}$, $m_1 \geqq m_2 \geqq 0$, where (m_1, m_2) is the heighest weight of $\lambda | SO(5)$ and ε is the image of $-1 \in O(5)$. Besides, $\tau(\lambda)$ corresponds to $(-m_2, -m_1)$.*

Theorem 2. 5 (Kashiwara-Vergne loc. cit., Lions-Vergne [21]). *For any $P(y)$, put $f_\infty(y) = P(yR) \exp(-2\pi \operatorname{tr}(yQ{}^t y))$, where Q is a positive definite symmetric matrix in $M_5(R)$ and $R\,{}^tR = Q$. Then, we have*

$$(R_{Q,\infty}(g) f_\infty)(y) = (t_\infty(g)\, m(g, i))^5\, P((Ci + D)^{-1}\, yR) \exp(2\pi i \operatorname{tr}(ZyQ{}^t y))$$

for all $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, *where* $Z = g(i)$.

2. 3. Now, we take a special Q for our purpose. Let B be a definite quaternion algebra over $\mathbb{Q}$ with discriminant d. We fix a basis (ω_i) $(i = 1 \dots 4)$ over $\mathbb{Z}$ of a maximal order O of B. We identify B with $\mathbb{Q}^4$ by this basis. On $\mathbb{Q}^5 \triangleq B \oplus \mathbb{Q}$, we define a quadratic form by $N(x) + t^2$ for $(x, t) \in B \oplus \mathbb{Q}$, where $N(x)$ is the reduced norm of B. The symmetric matrix attached to this form is given by:

$$Q = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \in M_5(Q), \quad \text{where} \quad S = \frac{1}{2} (\operatorname{tr}(\omega_i \bar{\omega}_j)) \qquad (i, j = 1 \dots 4),$$

and tr is the reduced trace. Q is obviously positive definite half-integral. We also identify $M_{2,5}(Q)$ with $(B \oplus \mathbb{Q})^2$. For $y \in M_{2,5}(\mathbb{Q})$, which is identified with

$${}^t((y_1, t_1), (y_2, t_2)) \in (B \oplus \mathbb{Q})^2,$$

we have

$$yQ{}^t y = \begin{pmatrix} N(y_1) + t_1^2 & t_1 t_2 + \dfrac{\operatorname{tr}(y_1 \bar{y}_2)}{2} \\ t_1 t_2 + \dfrac{\operatorname{tr}(y_1 \bar{y}_2)}{2} & N(y_2) + t_2^2 \end{pmatrix},$$

where $^-$ is the canonical involution of B. Put

$$G' = \{ g \in M_2(B);\ g\,{}^t\bar{g} = n(g)\, 1_2,\ n(g) \in \mathbb{Q}^\times \}.$$

Let G'_A be the adelization of G' and G'_v be the v-component. Let (κ, V_κ) be an irreducible representation of $Sp(2)$ which factors through $SO(5) \cong Sp(2)/\pm 1$. This means that the corresponding Young diagram (f_1, f_2) satisfies that $f_1 + f_2 = \text{even}$. Put $O_p = O \otimes \mathbb{Z}_p$, and $U_p = GL_2(P_p) \cap G'_p$. Put $U = G'_\infty \prod_p U_p$. Then, the space of automorphic forms belonging to U with weight κ is defined by:

$$M_{f_1, f_2}(U) = \{f : G'_A \to V_\kappa;\ f(ahu) = \kappa(u)^{-1} f(h) \quad \text{for all} \quad a \in G',\ u \in U,\ h \in G'_A\}$$

(cf. Hashimoto [5]). We denote by λ the representation of $SO(5)$ corresponding to κ. The Young diagram of λ is $\left(\frac{f_1+f_2}{2}, \frac{f_1-f_2}{2}\right)$. Take $\mathfrak{H}(\lambda)$ as in 2. 2. Let $\{P_i(y)\}$ $(i = 1 \dots \dim \mathfrak{H}(\lambda))$ be a basis of $\mathfrak{H}(\lambda)$. Put $f_{i,\infty}(y) = P_i(y) \exp(-2\pi \operatorname{tr}(yQ^t y))$. Let f_p be the characteristic function of $M_{2,5}(Z_p)$ for each p. Put $f_i = f_{i,\infty} \prod_p f_p \in S(X_A)$, where $X = M_{2,5}(Q)$. We have

$$X_A \triangleq (B_A \oplus Q_A)^2 \cong \left\{\begin{pmatrix} t & r \\ \bar{r} & -t \end{pmatrix};\ t \in Q_A,\ r \in B_A\right\}^2$$

as vector spaces over Q_A. For $h \in G'_A$ and $(X_1, X_2) \in X_A$, $\left(X_i = \begin{pmatrix} t_i & r_i \\ \bar{r}_i & -t_i \end{pmatrix}\right)$, put

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \rho(h) = \begin{pmatrix} h^{-1} X_1 h \\ h^{-1} X_2 h \end{pmatrix}.$$

This defines an action of G'_A on X_A. It is proved in the same way as in Yoshida [32] that, for a fixed $\bar{g} \in \bar{G}_A$, $\sum_{y \in X} (\pi(\bar{g}) f_i)(y\rho(h))$ is convergent and continuous on G'_A. We denote by $\langle\ ,\ \rangle$ a $(Sp(2))$ invariant metric on V_κ. We regard $\mathbb{R}^\times$ as a subgroup of G'_∞ by embedding $\mathbb{R}^\times \ni a \to \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G'_\infty$.

Definition 2. 6. For any $\varphi \in M_\kappa(U)$, we define a $\mathbb{C}$ valued function on $\bar{G}_A^+$ by

$$\Phi_i(\bar{g}) = \int_{\mathbb{R}^\times G' \backslash G'_A} \langle \sum_{y \in X} (\pi(\bar{g}) f_i)(y\rho(h)), \varphi(h) \rangle \, dh,$$

where dh is a Haar measure on G'_A. We put $\sigma(\varphi) = \Phi(\bar{g}) = (\Phi_i(\bar{g}))$ (column vector).

For our special Q of this subsection, N_p in (1. 2) is equal to the p-part of d', where d' is the least common multiple of d and 4. We put

$$\Gamma'_0(d') = G^+ \cap \prod_p K_p \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{Z});\ C \equiv 0 \bmod d',\ \det A \equiv 1 \bmod 4\right\}.$$

Theorem 2. 7. *Let κ be the representation of $Sp(2)$ which corresponds with (f_1, f_2) $(f_1 \geqq f_2 \geqq 0, f_1 + f_2 = \text{even})$. For any $\varphi \in M_k(U)$, $\sigma(\varphi) = \Phi$ is an automorphic form belonging to $\Gamma'_0(d')$ with weight* $\det^{\frac{f_1 - f_2 + 5}{2}} \otimes \operatorname{Sym}(f_2)$, *where* $\operatorname{Sym}(f_2)$ *is the symmetric tensor representation of $GL_2(C)$ of degree f_2.*

Proof. We define an action of G'_A on $f \in S(X_A)$ by: $(\rho(h) f)(y) = f(y\rho(h))$. Then, for any $\bar{g} \in \bar{G}_A^+$, we get

$$(2.8) \qquad \pi(\bar{g})\, \rho(h) = \rho(h)\, \pi(\bar{g}).$$

Then (2. 8) can be proved directly for the generators of $\bar{G}_p^+$ as in Proposition 1. 3. The key point is the fact that $\rho(h)\,Q\,{}^t\rho(h)=Q$. The assertion for $\bar{G}_A^+$ follows immediately from this. Now, we check each condition in Definition 2. 1.

(C1) G^+ is generated by $\begin{pmatrix}1 & 0\\ 0 & \lambda^2 1\end{pmatrix}$, $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}a & 0\\ 0 & {}^t a^{-1}\end{pmatrix}$, and J. For $\gamma\in G^+$ and $f'=\prod_v f'_v\in S(X_A)$, we have

$$\pi(\gamma)f'=s(\gamma)\prod_v R_{Q,v}(\gamma)\,s_v(\gamma)f'_v=\prod_v R_{Q,v}(\gamma)f'_v.$$

As for the first three generators, it is obvious that

$$\sum_{y\in X}\pi(\gamma)f'(y)=\sum_{y\in X}f'(y),$$

by virtue of the product formulae $\prod_v |*|_v=\prod_v \chi_v(*)=\prod_v \gamma_v(*)=1$. As for J, we get

$$\pi(J)f'=\int_{X_A} f'(y')\,\chi(-\mathrm{tr}(yQ\,{}^t y'))\,dy',$$

where $\chi=\prod_v \chi_v$. By virtue of (2. 8), we get

$$\rho(h)\,\pi(\gamma)f'=\int_{X_A}(\rho(h)f')(y')\,\chi(-\mathrm{tr}(yQ\,{}^t y'))\,dy'.$$

By the Poisson summation formula, we get

$$\sum_{y\in X}(\pi(\gamma)f')(y\rho(h))=\sum_{y\in X}f'(y\rho(h)).$$

If we put $f'=\pi(\bar{g})f$, we get the assertion.

The conditions (C2), (C3), (C5) are obvious.

(C4) is a direct consequence of Theorem 2. 5, noting that $k_\infty(i)=i$ for any $k_\infty\in K_\infty$ and the contragredient of $\left(-\frac{f_1-f_2}{2},\ -\frac{f_1+f_2}{2}\right)$ is $\left(\frac{f_1+f_2}{2},\ \frac{f_1-f_2}{2}\right)$. q.e.d.

We express now $\sigma(\varphi)$ in the classical language as in (2. 2). For $Z\in\mathfrak{H}_2$, take $g=\begin{pmatrix}A & B\\ 0 & {}^t A^{-1}\end{pmatrix}\in Sp(2,\mathbb{R})$ such that $\det A>0$ and $g(i1)=A\,{}^t A\,i+B\,{}^t A=Z$. Then, $\pi((g,1))f$ can be easily calculated. Take a double coset decomposition

$$G'_A=\coprod_{i=1}^{H} G' h_i U,$$

so that the ∞ components $(h_i)_\infty=1$. We take the Haar measure dh such that $\mathrm{vol}\,(Sp(2)\prod_p U_p)=1$. Put $\Gamma_i=h_i U h_i^{-1}\cap G'$ and $L=\prod_p M_{2,5}(\mathbb{Z}_p)\subset X_A$. Then, $f_j(Z)$ attached to the j-th component of $\sigma(\varphi)$ as in (2. 2) is given by:

$$\begin{aligned} f_j(Z)&=\int_{\mathbb{R}^\times G'\backslash G'_A}\Big\langle \sum_{y\in L\rho(h)^{-1}\cap X} P_j(y\rho(h))\exp(2\pi i\,\mathrm{tr}(ZyQ\,{}^t y)),\ \varphi(h)\Big\rangle\,dh\\ &=\sum_{i=1}^{H}\frac{1}{|\Gamma_i|}\Big\langle \sum_{y\in L\rho(h_i)^{-1}\cap X} P_j(y)\exp(2\pi i\,\mathrm{tr}(ZyQ\,{}^t y)),\ \varphi(h_i)\Big\rangle. \end{aligned}$$

For example, if κ is trivial, then

$$f_j(Z)=\sum_{i=1}^{H}\frac{\varphi(h_i)}{|\Gamma_i|}\sum_{y\in Lp(h_i)^{-1}\cap X}\exp\left(2\pi i\,\mathrm{tr}\,(ZyQ{}^ty)\right).$$

§ 3. Hecke theory

3. 1. We explain some general Hecke theory on $S(\Gamma,\det^{\frac{m}{2}}\otimes\tau)$. Take

$$Q={}^tQ\in M_m(\mathbb{Q})$$

as in § 1. Assume that Q is positive definite.

First, we need some lemma. We call a prime p a good prime if $p\neq 2$ and $N_p=1$, that is, if $K_p=GSp(n,\mathbb{Z}_p)\cap G_p^+$.

Lemma 3. 1.[1]) *For a good prime p and any $\bar{\omega}\in\bar{G}_p^+$, the double coset decomposition:*

$$\bar{K}_p\bar{\omega}\bar{K}_p=K_p\bar{\omega}K_p\cup K_p\bar{\omega}K_p(1,-1)$$

is disjoint.

Proof. We can assume that $\bar{\omega}=(\omega,1)$ and that ω is a diagonal matrix whose diagonal components are given by $(p^{e_1},\dots,p^{e_n},p^{f_1},\dots,p^{f_n})$ with $e_i+f_i=2\delta$ and $e_1\leqq e_2\leqq\cdots\leqq e_n\leqq f_n\leqq\cdots\leqq f_1$. Under this assumption, it is easy to see that $K_p\cap\omega K_p\omega^{-1}$ is spanned by "upper triangular" and "lower triangular" matrices. Now, let $h,k\in K_p$ be elements such that $h\omega=\omega k$. It is sufficient to prove that $\beta_p(h,\omega)=\beta_p(\omega,k)$, that is, $c_p(h,\omega)\,s_p(h)=c_p(\omega,k)\,s_p(k)$. Let f be the characteristic function of $M_{n,m}(\mathbb{Z}_p)$. Then, $R_{Q,p}(h)f=s_p(h)f$ by definition. On the other hand, we have

$$\begin{aligned}R_{Q,p}(h)\,R_{Q,p}(\omega)f&=c_p(h,\omega)\,R_{Q,p}(h\omega)f\\&=c_p(h,\omega)\,c_p(\omega,k)\,s_p(k)\,R_{Q,p}(\omega)f.\end{aligned}$$

So, we should prove that the actions of $R_{Q,p}(h)$ on f and $R_{Q,p}(\omega)f$ are same for any $h\in K_p\cap\omega K_p\omega^{-1}$. We may assume that h is "upper or lower triangular". When h is "upper triangular", the proof is a direct calculation, and we omit it here. Put

$$h=\begin{pmatrix}1&0\\x&1\end{pmatrix}\in K_p\cap\omega K_p\omega^{-1}.$$

If we denote ω by $\begin{pmatrix}a&0\\0&d\end{pmatrix}$, then, $d^{-1}xa\in M_n(\mathbb{Z}_p)$. Put $f'(y)=f({}^tap^{-\delta}y)$. Then, $R_{Q,p}(\omega)f=$ constant times f'. We have $\begin{pmatrix}1&0\\-x&1\end{pmatrix}=-J\begin{pmatrix}1&x\\0&1\end{pmatrix}J$. By direct calculation, we get $R_{Q,p}(J)f'=f_1\gamma_p(1)^{-nm}\,|\det Q|^{\frac{n}{2}}\,|p^{\delta n}\det a|^m$, where f_1 is the characteristic function of $p^{-\delta}aM_{n,m}(\mathbb{Z}_p)$, $R_{Q,p}\begin{pmatrix}1&x\\0&1\end{pmatrix}f_1=\chi_p(\mathrm{tr}\,(xyQ{}^ty)\,f_1(y)=f_1$, because, for $y=p^{-\delta}ay_0$,

[1]) A lemma similar to this one has been proved independently by Hayakawa in classical terminology.

$y_0 \in M_{n,m}(\mathbb{Z}_p)$, we have $\mathrm{tr}(xyQ^ty) = \mathrm{tr}(p^{-2\delta t}axay_0 Q^ty_0) = \mathrm{tr}(d^{-1}xay_0Q^ty_0) \in \mathbb{Z}_p$, and finally, we get $R_{Q,p}\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} f' = \mathrm{const} \times f'$, where the constant does not depend on y and ω. q.e.d.

We define a measure $d\bar{g}$ on $\bar{G}_p^+$ by putting

$$\int_{\bar{G}_p^+} \varphi(\bar{g})\, d\bar{g} = \int_{G_p^+} \big(\varphi(g, 1) + \varphi(g, -1)\big)\, dg,$$

where φ is a function on $\bar{G}_p^+$ and dg is a Haar measure on G_p^+ such that $\mathrm{vol}(K_p) = \frac{1}{2}$. Then, $d\bar{g}$ is $\bar{G}_p^+$ invariant. Let ψ_i $(i = 1, 2)$ be continuous functions on $\bar{G}_p^+$ such that $\psi_i(\bar{g}(1, -1)) = -\psi_i(\bar{g})$, one of which is of compact support. Then, the product is defined by:

$$\psi_1 * \psi_2 = \int_{\bar{G}_p^+} \psi_1(\bar{h}\bar{g}^{-1})\, \psi_2(\bar{g})\, d\bar{g}.$$

By virtue of Lemma 3. 1, for any $\omega \in \bar{G}_p^+$, we can define a function $\psi(\omega)$ by:

$$\psi(\omega)(\bar{g}) = \begin{cases} \zeta & \text{if } \bar{g} \in K_p\omega K_p(1, \zeta), \\ 0 & \text{otherwise,} \end{cases}$$

where $\zeta = \pm 1$. The action of a double coset $\bar{K}_p\omega\bar{K}_p$ on a function $\Phi \in S(\Gamma, \det^{\frac{m}{2}} \otimes \tau)$ is defined by $c(\omega)\big((\omega) * \Phi\big)$, where $c(\omega)$ is some normalizing constant which will be chosen later. It is easy to see that

$$([\bar{K}_p\omega\bar{K}_p]\, \Phi)(\bar{g}) = c(\omega) \sum_{i=1}^{u} \Phi(\bar{g}\bar{g}_i^{-1}),$$

where the summation runs through a set of representatives of the left cosets of

$$K_p\omega K_p = \coprod_{i=1}^{u} K_p\bar{g}_i.$$

For Φ, define $f(Z)$ as in (2. 2). For the reader's convenience, we write here how to calculate the action of the Hecke operators on $f(Z)$. Take $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ such that $g(i) = Z$. We denote $(g, 1) \in \bar{G}^+$ also by g. Put $f_i(Z) = J(g, i)^m\, \tau(Ci + D)\, \Phi(g\bar{g}_i^{-1})$. Then, by definition, $[K_p\omega K_p] f = c(\omega) \sum_{i=1}^{u} f_i$. Put $\bar{g}_i = (g_i, \zeta_i)$. Put $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} = h_i\omega k_i$ for some $h_i, k_i \in K_p$. We may take g_i in $G^+ \cap \prod_{q \neq p} K_p$.

Theorem 3. 2. *Notations and assumptions being as above, we get*

$$f_i(Z) = s_p(\omega)\,(m(g_i, Z)\,t_\infty(g_i))^m\,s_p(h_i)\,s_p(k_i^{-1})$$
$$c_p(g_i, k_i^{-1})\,c_p(h, \omega) \prod_{q \neq p} s_q(g_i)\,\tau(C_i\sqrt{-1} + D_i)^{-1} f(g_i Z),$$

where $g_i = \lambda s_i$, $\lambda \in R$, $\lambda > 0$, $s_i \in Sp(2, \mathbb{R})$ *and* $s_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$.

This theorem will not be used in the rest of this paper, and the proof will be omitted here. Actually, we can take g_i so that it is "upper triangular", and all the quantities in the above theorem can be explicitly calculated at least when ω is given. Explicit actions of Hecke operators have been calculated by several mathematicians independently, e.g. Juravlev [14], [15] [2]), for general genus, and Hina, Hayakawa, and the present author for genus two. During the preparation of this paper, the author had contact with Hina, some of whose results convinced the author that we should take symplectic similitudes group with only square multiplicators. The author would like to thank him for this point.

3. 2. Now we go back to our special case in § 2. 3. We define the normalizing factor of the Hecke operators as follows. Take a representation of $Sp(2)$ with $f_1 + f_2 =$ even. Take $h_0 \in G_p'$ and denote the multiplicator of h_0 by p^δ (i.e. $h_0{}^t\bar{h}_0 = p^\delta 1_2$). Take h_s so that

$$U h_0 U = \coprod_{s=1}^{v} h_s U \text{ (disjoint)}.$$

Then, for $\varphi \in M_\kappa(U)$, we define

$$(3.3)\qquad (T(U h_0 U)\,\varphi)(h) = p^{\delta\frac{f_1+f_2}{2}} \sum_{s=1}^{v} \kappa(h_s)\,\varphi(h h_s).$$

On the other hand, take $\omega \in G_p^+$ and denote by $p^{2\delta}$ the similitude of ω. Take $\bar{g}_i \in \bar{G}_p^+$ so that

$$K_p(\omega, 1)\,K_p = \coprod_{i=1}^{u} K_p \bar{g}_i \text{ (disjoint)}.$$

For $\Phi \in S(\Gamma_0'(d'), \det^{\frac{f_1 - f_2 + 5}{2}} \otimes \mathrm{Sym}(f_2))$, we define

$$(3.4)\qquad (T_0(\bar{K}_p \omega \bar{K}_p)\,\Phi)(\bar{g}) = p^{\delta\frac{f_1+f_2-1}{2}} \sum_{i=1}^{u} \Phi(\bar{g}\bar{g}_i^{-1}).$$

More explicitly, this action can be described as follows. Put

$$\bar{g}_i = (g_i, \zeta_i) = (h, 1)\,(\omega, 1)\,(k, 1)$$

for some $h, k \in K_p$. Put $\bar{g}_i^{-1} = (g_i^{-1}, \zeta_i')$. Then, $\zeta_i' = \beta_p(g_i, g_i^{-1})\,\beta_p(h, \omega)\,\beta_p(h\omega, k)$. On the other hand, for $f \in S(X_p)$, we get

$$\pi(\bar{g}_i^{-1}) f = \zeta_i' s_p(g_i^{-1})\,R_p(g_i^{-1}) f.$$

[2]) The author was informed by Prof. Böcherer of these papers after he finished this work. He would like to express his thanks.

Denote $\zeta_i' s_p(g_i)$ by $\varepsilon(g_i)$. Then, we get

$$\varepsilon(g_i)=s_p(g_i)\, s_p(g_i^{-1})\, s_p(h)\, s_p(\omega)\, s_p(h\omega)\, s_p(h\omega)\, s_p(k)\, s_p(g_i) \times c_p(g_i, g_i^{-1})\, c_p(h, \omega)\, c_p(h\omega, k)\, s_p(g_i^{-1}).$$

Thus,

$$\varepsilon(g_i)=s_p(\omega)\, c_p(g_i, g_i^{-1})\, c_p(h, \omega)\, c_p(g_i, k^{-1})\, s_p(h)\, s_p(k^{-1}). \tag{3. 5}$$

Here, we have some ambiguity on $s_p(\omega)$, because it was arbitrarily chosen in § 1. From now on, for the sake of simplicity, we put $s_p(\omega)=1$. When

$$\omega, \text{ or } h_0=\begin{pmatrix} p^{e_1} & 0 & 0 & 0 \\ 0 & p^{e_2} & 0 & 0 \\ 0 & 0 & p^{f_1} & 0 \\ 0 & 0 & 0 & p^{f_2} \end{pmatrix},$$

we write $T_0(K_p\omega K_p)=T_0(p^{e_1}, p^{e_2}, p^{f_1}, p^{f_2})$ and $T(Uh_0U)=T(p^{e_1}, p^{e_2}, p^{f_1}, p^{f_2})$.

Our aim of this subsection is to prove the following two key theorems.

Theorem 3. 6. *Take a prime p such that $p \nmid d'$. Take disjoint coset decomposition as follows:*

$$U_p\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}U_p=\coprod_{s=1}^{v} U_p h_s \qquad (h_s \in G_p'),$$

$$K_p\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}K_p=\coprod_{i=1}^{u} K_p g_i \qquad (g_i \in G_p^+).$$

Let f_p be the characteristic function of $M_{2,5}(\mathbb{Z}_p)$. Then,

$$\varepsilon_p \sum_{i=1}^{u} \varepsilon(g_i)\, (R_p(g_i^{-1}) f_p)\, (Y)=\sqrt{p} \sum_{s=1}^{v} f_p(Y\rho(h_s)^{-1}) \tag{3. 7}$$

for all $Y \in M_{2,5}(\mathbb{Q}_p)$, where $\varepsilon_p=1$ or $\sqrt{-1}$, according as $p \equiv 1$ or $3 \bmod 4$, respectively.

Theorem 3. 8. *We use the same notations as above, but this time, we take the following double cosets:*

$$U_p\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}U_p=\coprod_{s=1}^{v} U_p h_s, \quad K_p\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}K_p=\coprod_{i=1}^{u} K_p g_i.$$

Then, we have

$$(3.9) \qquad \sum_{i=1}^{u} \varepsilon(g_i)\,(R_p(g_i^{-1})f_p)(Y) = p \sum_{s=1}^{v} f_p(Y\rho(h_s)^{-1}).$$

In the rest of this section, we shall give the proof of these theorems. First, we give $\varepsilon(g_i)$. Put

$$R(p) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} SL_2(Z);\ x = 0, \dots, p-1 \right\}$$

and

$$R(p^2) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} py, & -1 \\ 1, & 0 \end{pmatrix} SL_2(Z);\ x = 0, \dots, p^2-1,\ y = 0, \dots, p-1 \right\}.$$

Proposition 3.10. *The set of g_i in Theorem 3.6 can be chosen to be the set of following elements of type (1), (2) and (3).*

$$(1) \qquad \begin{pmatrix} p & 0 & a & b \\ 0 & p & b & c \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \text{ where } 0 \leqq a, b, c \leqq p-1, \text{ and } \begin{pmatrix} a & b \\ b & c \end{pmatrix} \equiv U \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} {}^tU \bmod p$$

for some $U \in GL_2(\mathbf{F}_p)$ and $f \in \mathbf{F}_p^\times$,

$$(2) \qquad \begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & U \end{pmatrix}, \qquad U \in R(p),$$

$$(3) \qquad \begin{pmatrix} p & 0 & 0 & pb \\ 0 & 1 & b & c \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & U \end{pmatrix}, \qquad U \in R(p^2), \quad 0 \leqq b \leqq p-1, \quad 0 \leqq c \leqq p^2-1.$$

If g_i is of type (1), then $\varepsilon(g_i) = \left(\frac{-f}{p}\right)$, and if g_i is of type (2) or (3), then $\varepsilon(g_i) = \left(\frac{-1}{p}\right)$, where $\left(\frac{*}{p}\right)$ is the Legendre symbol.

Proposition 3. 11. *The set of g_i in Theorem* 3. 8 *can be chosen to be the set of following elements of type* (1),..., (6).

$$(1)\quad \begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(2)\quad \begin{pmatrix} p & 0 & a & b \\ 0 & p & b & c \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \qquad \det\begin{pmatrix} a & b \\ b & c \end{pmatrix} \not\equiv 0 \bmod p, \quad 0 \leqq a, b, c \leqq p-1,$$

$$(3)\quad \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & c \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix} \qquad 0 \leqq a, b, c \leqq p^2-1,$$

$$(4)\quad \begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & U \end{pmatrix} \qquad 1 \leqq y \leqq p-1, \quad U \in R(p),$$

$$(5)\quad \begin{pmatrix} p & 0 & x & py \\ 0 & 1 & y & z \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & U \end{pmatrix} \qquad 1 \leqq x \leqq p-1, \quad 0 \leqq y, \ z \leqq p-1, \quad U \in R(p),$$

$$(6)\quad \begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & U \end{pmatrix} \qquad 0 \leqq x \leqq p^2-1, \quad U \in R(p^2).$$

For each g_i of the above type (1),..., (6), $\varepsilon(g_i)$ *is given by*

$$1, \ \left(\frac{b^2-ac}{p}\right), \ 1, \ \left(\frac{-y}{p}\right), \ \left(\frac{-x}{p}\right), \ \textit{or } 1, \ \textit{respectively}.$$

For $Y \in M_{2,5}(\mathbb{Q}_p) \cong (B_p \oplus \mathbb{Q}_p)^2$, we put $Y = {}^t((y_1, t_1), (y_2, t_2))$, $y_i \in B_p$, $t_i \in \mathbb{Q}_p$. We define T_i $(i=1,2,3)$ and D as follows:

$$T_1 = N(y_1) + t_1^2, \quad T_2 = \mathrm{tr}(y_1 \bar{y}_2) + 2t_1 t_2, \tag{3. 12}$$
$$T_3 = N(y_2) + t_2^2, \quad D = 4T_1 T_3 - T_2^2.$$

In other words, if we identify Y with $(Y_1, Y_2) \in \left\{ \begin{pmatrix} t & r \\ \bar{r} & -t \end{pmatrix} ; \ t \in \mathbb{Q}_p, \ r \in B_p \right\}^2$, we get $(T_1, T_2, T_3) = \left(-\det Y_1, \dfrac{\mathrm{tr}(Y_1 Y_2)}{2}, -\det Y_2 \right)$, and it is obvious that T_i and D are invariant by mappings $Y \to Y\rho(h)$ $(h \in G'_p)$.

Proposition 3. 13. *For any $Y \in M_{2,5}(\mathbb{Q}_p)$, the left-hand side of* (3. 7) *is given by the summation of the following quantities*:

(1)
$$f_p(Y) \times \begin{cases} p\sqrt{p}\left(\dfrac{T_1}{p}\right), & \text{if } p \mid T_3,\ p \mid D, \\ p\sqrt{p}\left(\dfrac{T_3}{p}\right), & \text{if } p \nmid T_3,\ p \mid D, \\ 0, & \text{if } p \nmid D, \end{cases}$$

(2)
$$p^{\frac{5}{2}} \sum_{U \in R(p)} f_p\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} UY \right),$$

(3)
$$p^{\frac{1}{2}} f_p\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y \right) \times \begin{cases} 1 & \text{if } T_1, T_2 \in \mathbb{Z}_p \\ 0 & \text{otherwise} \end{cases}$$
$$+ p^{\frac{1}{2}} \sum_{x=0}^{p-1} f_p\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} Y \right) \times \begin{cases} 1 & \text{if } T_2, T_3 \in \mathbb{Z}_p, \\ 0 & \text{otherwise}. \end{cases}$$

Each quantity labelled (*n*) (*n* = 1, 2, 3) *is the contribution of cosets of type* (*n*) *in Proposition* 3. 10.

Proposition 3. 14. *For any $Y \in M_{2,5}(\mathbb{Z}_p)$, the left-hand side of* (3. 9) *is given by the summation of the following quantities*:

(1) $p^5 f_p(p^{-1} Y)$,

(2) $f_p(Y) \times \begin{cases} p(p-1), & \text{if } p \mid D, \\ -p, & \text{if } p \nmid D, \end{cases}$

(3) $f_p(pY) \times \begin{cases} p, & \text{if } T_1, T_2, T_3 \in \mathbb{Z}_p, \\ 0 & \text{otherwise}, \end{cases}$

(4) $$p^3\left(\frac{T_1}{p}\right)f_p\left(\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}Y\right)+p^3\left(\frac{T_3}{p}\right)\sum_{x=0}^{p-1}f_p\left(\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix}Y\right),$$

(5) $$pf_p\left(\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}Y\right)\times\begin{cases}\left(\dfrac{T_3}{p}\right) & \text{if } T_1, T_2\in \mathbb{Z}_p,\\ 0 & \text{otherwise},\end{cases}$$

$$+p\sum_{x=0}^{p-1}f_p\left(\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix}Y\right)\times\begin{cases}\left(\dfrac{T_1+xT_2+x^2T_3}{p}\right) & \text{if } T_2, T_3\in \mathbb{Z}_p,\\ 0 & \text{otherwise},\end{cases}$$

(6) $$p^2\sum_{x=0}^{p-1}f_p\left(\begin{pmatrix}p^{-1}&0\\0&p\end{pmatrix}\begin{pmatrix}px&1\\-1&0\end{pmatrix}Y\right)\times\begin{cases}1, & \text{if } T_1\in \mathbb{Z}_p,\\ 0 & \text{otherwise},\end{cases}$$

$$+p^2\sum_{x=0}^{p^2-1}f_p\left(\begin{pmatrix}p^{-1}&0\\0&p\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix}Y\right)\times\begin{cases}1, & \text{if } T_3\in \mathbb{Z}_p,\\ 0 & \text{otherwise}.\end{cases}$$

Each quantity labelled (*n*) ($n=1,\ldots,6$) *is the contribution of cosets of type* (*n*) *in Proposition* 3.11.

Proof of Propositions 3.10 *and* 3.11. As we have chosen g_i so that it is "upper triangular", we can calculate $R_p(g_i^{-1})f_p$ directly from the definition. For example, put

$$g=\begin{pmatrix}p&0&a&b\\0&p&b&c\\0&0&p&0\\0&0&0&p\end{pmatrix},$$

$$\alpha=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&p^{-2}&0\\0&0&0&p^{-2}\end{pmatrix},\quad \beta=\begin{pmatrix}p^{-1}&0&0&0\\0&p^{-1}&0&0\\0&0&p&0\\0&0&0&p\end{pmatrix}\quad \gamma=\begin{pmatrix}1&0&\begin{pmatrix}-a&-b\\-b&-c\end{pmatrix}p^{-1}\\0&1&\\0&0&0\quad 0\\0&0&0\quad 0\end{pmatrix}.$$

Then, $g^{-1}=\alpha\beta\gamma$, and we get

$$(R_p(\gamma)f_p)(Y)=\exp(2\pi ip^{-1}(aT_1+bT_2+cT_3))f_p(Y),$$
$$(R_p(\beta)R_p(\gamma)f_p)(Y)=\exp(2\pi ip^{-3}(aT_1+bT_2+cT_3))f_p(p^{-1}Y)p^5,$$

and

$$(R_p(\alpha)R_p(\beta)R_p(\gamma)f_p)(Y)=\exp(2\pi ip^{-1}(aT_1+bT_2+cT_3))f_p(Y).$$

By the calculation of 2-cocycles, we have $R_p(\alpha)\,R_p(\beta)\,R_p(\gamma)=R_p(\alpha\beta\gamma)$. Now, we calculate the summation of

$$I(a,b,c)=\left(\frac{-f}{p}\right)\exp\left(2\pi i p^{-1}(aT_1+bT_2+cT_3)\right)f_p(Y)$$

for all g of type (1) in Proposition 3. 10. If $Y\in M_{2,5}(\mathbb{Z}_p)$, then this is zero, so we can assume that $T_i\in\mathbb{Z}_p$ $(i=1,2,3)$. We have $ac-b^2\equiv 0 \bmod p$. If $a=0$, then $b=0$ and $c\equiv -f \bmod p$. The partial sum of $I(a,b,c)$ over these elements is given by:

$$\sum_{c=1}^{p-1}\left(\frac{-c}{p}\right)\exp(2\pi i p^{-1}cT_3)=\begin{cases}\left(\dfrac{-T_3}{p}\right)\varepsilon_p\sqrt{p}, & \text{if } p\nmid T_3,\\ 0, & \text{if } p\mid T_3.\end{cases}$$

If $a\neq 0$, then $c\equiv b^2a^{-1} \bmod p$ and $f\equiv a \bmod p$. Then, the partial sum of $I(a,b,c)$ over these elements is given by:

$$\begin{aligned}I&=\sum_{b=0}^{p-1}\sum_{a=1}^{p-1}I(a,b,c)\\&=\sum_{b=0}^{p-1}\sum_{a=1}^{p-1}\left(\frac{-a}{p}\right)\exp\left(2\pi i p^{-1}\left(\frac{1}{aT_3}\left(T_3b+\frac{aT_2}{2}\right)^2+\frac{a(4T_1T_3-T_2^2)}{4T_3}\right)\right).\end{aligned}$$

We may regard every element a, b, etc. as an element of $\mathbf{F}_p$. If $p\nmid T_3$, then $T_3b+2^{-1}aT_2$ runs through all elements of $\mathbf{F}_p$, and we get

$$I=\sum_{a=1}^{p-1}\varepsilon_p\sqrt{p}\left(\frac{-T_3}{p}\right)\exp\left(\frac{Da}{4T_3}\right).$$

In this case, we get

$$I=\begin{cases}(p-1)\,\varepsilon_p\sqrt{p}\left(\dfrac{-T_3}{p}\right), & \text{if } p\mid D,\\ \varepsilon_p\sqrt{p}\left(\dfrac{-T_3}{p}\right), & \text{if } p\nmid D.\end{cases}$$

If $p\mid T_3$, it is easy to see that

$$I=\begin{cases}p^{\frac{3}{2}}\varepsilon_p\left(\dfrac{-T_1}{p}\right), & \text{if } p\mid T_2,\\ 0, & \text{if } p\nmid T_2.\end{cases}$$

Thus, combining above calculations, we get (1) of Proposition 3. 13. The other cases can be proved by more or less similar routine calculation, and the proof will be omitted here.

q.e.d.

Now, we must evaluate the right hand side of (3. 7) and (3. 9). First, we calculate $Y\rho(h_s)$ for $Y \in M_{2,5}(\mathbb{Q}_p)$. Define an injection j of $M_2(\mathbb{Q}_p) \oplus \mathbb{Q}_p$ into $M_4(\mathbb{Q}_p)$ by:

$$j\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) = \begin{pmatrix} t & x & 0 & y \\ w & -t & -y & 0 \\ 0 & z & t & w \\ -z & 0 & x & -t \end{pmatrix}.$$

We have $Y\rho(h_s)^{-1} = {}^t(j^{-1}(h_s j(Y_1) h_s^{-1}), j^{-1}(h_s j(Y_2) h_s^{-1}))$, where

$$Y = {}^t(Y_1, Y_2) \in (M_2(\mathbb{Q}_p) \oplus \mathbb{Q}_p)^2 \quad \text{and} \quad h_s \in G'_p \cong GSp(2, \mathbb{Q}_p).$$

For the sake of simplicity, we sometimes write an element

$$A = \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) \in M_2(\mathbb{Q}_p) \oplus \mathbb{Q}_p \text{ or } j(A)$$

by a vector (x, y, z, w, t).

Lemma 3. 15 (Andrianov [1]). *For* $h_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$, *a set of* $\{h_s\}$ *such that* $Uh_0U = \coprod_{s=1}^{v} Uh_s$ *(disjoint) is given by the following elements*:

(1) $\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

(2) $\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix}$, *where* $0 \leqq a \leqq p-1$, *and*

(i) $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *or* (ii) $U = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$, $0 \leqq q \leqq p-1$,

(3) $\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, *where* $0 \leqq a, b, c \leqq p-1$.

Lemma 3. 16. *When h_s is one of the above elements of type* (1), (2) (i) (ii), *or* (3), $h_s A h_s^{-1}$ *for* $A=(x, y, z, w, t)$ *is given respectively as follows*:

(1) $(x, py, p^{-1}z, w, t)$,

(2) (i) $\left(-pw, y+aw, z, -\dfrac{x+az}{p}, -t\right)$, $0\leqq a\leqq p-1$,

(ii) $\left(px, y-ax, z, \dfrac{w-q^2x-az-2qt}{p}, t+qx\right)$, $0\leqq a\leqq p-1$, $0\leqq q\leqq p-1$,

(3) $\left(x+az, \dfrac{y-cx+(b^2-ac)z+aw-2bt}{p}, pz, w-cz, t-bz\right)$,

$0\leqq a, b, c\leqq p-1$.

Proof. The proof is a direct calculation, and we omit it here.

Lemma 3. 17 (Andrianov [1]). *For* $h_0=\begin{pmatrix}1&0&0&0\\0&p&0&0\\0&0&p^2&0\\0&0&0&p\end{pmatrix}$, *a set of* $\{h_s\}$ *such that* $Uh_0U=\coprod_{s=1}^{v} Uh_s$ *is given by the following elements*:

(1) $\begin{pmatrix}p&0&\multicolumn{2}{c}{B}\\0&p&&\\0&0&p&0\\0&0&0&p\end{pmatrix}$, $B={}^tU\begin{pmatrix}f&0\\0&0\end{pmatrix}U$, $1\leqq f\leqq p-1$,

where (i) $U=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, *or* (ii) $U=\begin{pmatrix}1&q\\0&1\end{pmatrix}$, $0\leqq q\leqq p-1$,

(2) $\begin{pmatrix}p^2&0&0&0\\0&p&0&0\\0&0&1&0\\0&0&0&p\end{pmatrix}\begin{pmatrix}{}^tU^{-1}&0\\0&U\end{pmatrix}$,

where (i) $U=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, *or* (ii) $U=\begin{pmatrix}1&q\\0&1\end{pmatrix}$, $0\leqq q\leqq p-1$,

(3) $\begin{pmatrix}p&0&0&pb\\0&1&b&c\\0&0&p&0\\0&0&0&p^2\end{pmatrix}\begin{pmatrix}{}^tU^{-1}&0\\0&U\end{pmatrix}$, $0\leqq b\leqq p-1$, $0\leqq c\leqq p^2-1$,

where (i) $U=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, *or* (ii) $U=\begin{pmatrix}1&q\\0&1\end{pmatrix}$, $0\leqq q\leqq p-1$.

Lemma 3. 18. *When h_s is one of the above elements of type* (1) (i) (ii), 2 (i) (ii), *or* (3) (i) (ii), $h_s A h_s^{-1}$ *for* $A=(x, y, z, w, t)$ *is given respectively as follows*:

(1) (i) $(x, y-p^{-1}fx, z, w-p^{-1}fz, t)$, $1 \leqq f \leqq p-1$,

(ii) $\bigl(x+p^{-1}fz, y+p^{-1}f(-q^2x+w-2qt), z, w-p^{-1}fq^2z, t-p^{-1}fqz\bigr)$,

$1 \leqq f \leqq p-1$, $0 \leqq q \leqq p-1$,

(2) (i) $(-pw, py, p^{-1}z, -p^{-1}x, -t)$,

(ii) $\bigl(px, py, p^{-1}z, p^{-1}(-q^2x+w-2qt), t+qx\bigr)$, $0 \leqq q \leqq p-1$,

(3) (i) $\bigl(-pw, p^{-1}(y+b^2z+cw+2bt), pz, -p^{-1}(x+zc), -t-bz\bigr)$,

$0 \leqq b \leqq p-1$, $0 \leqq c \leqq p^2-1$,

(ii) $\bigl(px, p^{-1}(-cx-2bqx+y+b^2z-2bt), pz, p^{-1}(-q^2x-cz+w-2qt), t+qx-b\dot{z}\bigr)$,

$0 \leqq b, q \leqq p-1$, $0 \leqq c \leqq p^2-1$.

Proof. The proof is a direct calculation, and we omit it here.

Now, we give two preliminary remarks to (3. 7) and (3. 9).

Lemma 3. 19. *The both sides of* (3. 7) *and* (3. 9) *remain unchanged, even if we replace* $Y={}^t(Y_1, Y_2) \in \bigl(M_2(\mathbb{Q}_p) \oplus \mathbb{Q}_p\bigr)^2$ *by* ${}^t(Y_2, Y_1)$.

Proof. This is obvious for the right hand sides. As for the left hand side, put $I=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then, $I \in K_p$, as we assumed that $p \nmid d'$. The action of the Hecke operators does not depend on the choice of the representatives of the left K_p cosets, so we have

$$\sum_{i=1}^{u} \varepsilon(g_i)\, R_p(g_i^{-1}) f_p = \sum_{i=1}^{u} \varepsilon(g_i I)\, R_p(I g_i^{-1}) f_p = \sum_{i=1}^{u} \varepsilon(I g_i)\, c_p(I, g_i^{-1})\, R_p(I)\, R_p(g_i^{-1}) f_p.$$

By virtue of (3. 5) we have

$$\begin{aligned}\varepsilon(I g_i)\, c_p(I, g_i^{-1}) &= c_p(I, g_i^{-1})\, c_p(g_i I, I g_i^{-1})\, c_p(g_i I, I k^{-1})\, s_p(I k^{-1})\, s_p(h)\, c_p(h, \omega)\\ &= c_p(I, k^{-1})\, s_p(I k^{-1})\, s_p(k^{-1})\, \varepsilon(g_i)\\ &= s_p(I)\, \varepsilon(g_i) = \varepsilon(g_i),\end{aligned}$$

because $s_p(I)=1$ by definition. But for any $f' \in S(X_p)$, we have

$$\bigl(R_p(I) f'\bigr)\bigl({}^t(Y_1, Y_2)\bigr) = f'\bigl({}^t(Y_2, Y_1)\bigr), \quad Y_1, Y_2 \in \mathbb{Q}_p^5. \qquad \text{q.e.d.}$$

Lemma 3. 20. *Define* T_i $(i=1, 2, 3)$ *as in* (3. 12). *If* $T_i \notin \mathbb{Z}_p$ *for some* i, *then the both sides of* (3. 7) *and* (3. 9) *are zero.*

Proof. As we have already written, T_i is invariant by $Y \to Y\rho(h)$ $(h \in G'_p)$. So, this lemma is obvious for the right-hand side. Now, we shall prove that each quantity labelled (n) in Proposition 3. 13 or in Proposition 3. 14 is zero, if some T_i is not integral. This is obvious for (1), (2) in Proposition 3. 13, and (1), (2), (3), (4) in Proposition 3. 14. Now, we treat (3) in Proposition 3. 13. If $T_2 \notin \mathbb{Z}_p$, it is zero. If $T_3 \notin \mathbb{Z}_p$ and $T_2 \notin \mathbb{Z}_p$, then $Y_2 \in \mathbb{Z}_p^5$ and the quantity is zero. If $T_1 \notin \mathbb{Z}_p$ and the quantity (3) is not zero, then we have $T_2, T_3 \in \mathbb{Z}_p$ and $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} Y \in M_{2,5}(\mathbb{Z}_p)$ for some $x \in \mathbb{Z}_p$. So, $Y={}^t(U-p^{-1}xV, p^{-1}V)$ for some $U, V \in \mathbb{Z}_p^5$. If we identify Z_5 with

$$\left\{\begin{pmatrix} t & r \\ \bar{r} & -t \end{pmatrix};\ t \in \mathbb{Z}_p,\ r \in 0_p \cong M_2(\mathbb{Z}_p)\right\},$$

then $T_2 = 2^{-1}\operatorname{tr}(p^{-1}UV) - xT_3$. So, $\dfrac{\operatorname{tr}(p^{-1}UV)}{2} \in \mathbb{Z}_p$. On the other hand, we have

$$T_1 = \det(U-p^{-1}xV) = \det U - 2^{-1}x\operatorname{tr}(p^{-1}UV) + x^2\det(p^{-1}V) \in \mathbb{Z}_p,$$

which is a contradiction. Thus, this case is proved. The proof for (5) in Proposition 3. 14 is completely the same.

Now, we treat (6) in Proposition 3. 14. Assume that it is not zero. Then, we have $Y={}^t(-p^{-1}V, pU+xV)$ and $T_1 \in \mathbb{Z}_p$ or $Y={}^t(pU+p^{-1}xV, p^{-1}V)$ and $T_3 \in \mathbb{Z}_p$, for some $x \in \mathbb{Z}_p$ and $U, V \in \mathbb{Z}_p^5$. We can see from this that $T_1, T_3 \in \mathbb{Z}_p$. We get

$$T_2 = 2^{-1}\operatorname{tr}(p^{-1}V(pU+xV))$$

or $2^{-1}\operatorname{tr}((pU+p^{-1}xV)p^{-1}V)$. As $\operatorname{tr}(V^2) = -2\det V$, we can conclude that $T_2 \in \mathbb{Z}_p$.

q.e.d.

Proof of Theorems 3. 6 *and* 3. 8. The proof consists of rather routine elementary number theoretical calculations, but is very long. So, we sketch here only the outline of the proof. For the sake of simplicity, we denote by (L 1) (resp. (R 1)) the left (resp. right) hand side of (3. 7). Similarly, we denote by (L 2) (resp. (R 2)) the left (resp. right) hand side of (3. 9).

$$Y={}^t(Y_1, Y_2) \in (M_2(\mathbb{Q}_p) \oplus \mathbb{Q}_p)^2,\quad Y_1 = \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right),\quad Y_2 = \left(\begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix}, t'\right).$$

By Lemma 3. 20, we can assume that $T_i \in \mathbb{Z}$ $(i=1, 2, 3)$ in the following proof. But $M = M_2(\mathbb{Z}_p) \oplus \mathbb{Z}_p$. By virtue of Lemma 3. 19, we may divide the cases as follows:

(0) $Y_1 \notin p^{-1}M$ or $Y_2 \notin p^{-1}M$,

(I) $Y_i \in pM$ for $i=1, 2$,

(II) $Y_1 \in pM$, and $Y_2 \in M$, $Y_2 \notin pM$,

(III) $Y_1 \in pM$, and $Y_2 \in p^{-1}M$, $Y_2 \notin M$,

(IV) $Y_1 \in M$, $Y_1 \notin pM$, and $Y_2 \in M$, $Y_2 \notin pM$,

(V) $Y_1 \in M$, $Y_1 \notin pM$, and $Y_2 \in p^{-1}M$, $Y_2 \notin M$,

(VI) $Y_i \in p^{-1}M$, $Y_i \notin M$, for $i=1, 2$.

We calculate both sides of (3. 7) and (3. 9) in each case by using Proposition 3. 13, 3. 14 and Lemma 3. 16, 3. 18.

Case (0). Both sides are clearly zero, and the theorems are obvious.

Case (I). Every vector in Lemma 3. 16 and 3. 18 belongs to M and we have

$$(\mathrm{R}\,1)=(p+1)(p^2+1)p^{\frac{1}{2}} \quad \text{and} \quad (\mathrm{R}\,2)=p^2(p+1)(p^2+1).$$

In this case, we have $T_i \equiv 0 \bmod p$ $(i=1, 2, 3)$. We get

$$(\mathrm{L}\,1)=p^{\frac{5}{2}}(p+1)+p^{\frac{1}{2}}(p+1)=(\mathrm{R}\,1) \quad \text{and} \quad (\mathrm{L}\,2)=p^5+p(p-1)+p^3+p^4=(\mathrm{R}\,2).$$

Thus, the theorems are proved in this case.

Case (II). We have $T_1, T_2 \in p\mathbb{Z}_p$, and as we have assumed $T_3 \in \mathbb{Z}_p$, we get $D \in p\mathbb{Z}_p$. For $U \in R(p)$, we get $\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} UY \in M_{2,5}(\mathbb{Z}_p)$ if and only if $U=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So, we get

$$(\mathrm{L}\,1)=p^{\frac{1}{2}}\left(p\left(\frac{T_3}{p}\right)+p^2+p+1\right).$$

To obtain (R 1), we must treat various subcases. For the sake of simplicity, we denote vectors (with parameters) of type (1), (2) (i), (2) (ii), or (3) of Lemma 3. 16 by v_1, v_2, v_2', or v_3, respectively. By $C(v_2)$, we denote the set of parameters in v_2 such that $v_2 \in \mathbb{Z}_p^5$, and so on.

Subcase (a). Assume that $z' \in \mathbb{Z}_p^\times$. Then, $v_1 \notin \mathbb{Z}_p^5$. We have

$C(v_2)=\{a;\ x'+az' \equiv 0 \bmod p\}$ and $\#(C(v_2))=1$,

$C(v_2')=\left\{(a, q);\ 0 \leqq q \leqq p-1,\ a \equiv \dfrac{w'-2qt'-q^2x'}{z'} \bmod p\right\}$ and $\#(C(v_2'))=p$,

$C(v_3)=\{(a, b, c);\ b^2z'-2bt'+y'-cx'+aw'-acz' \equiv 0 \bmod p\}$.

The condition in $C(v_3)$ is a quadratic equation of b, and the discriminant is

$$D(a, c)=t'^2-z'(y'-cx'+aw'-acz').$$

For fixed a, c, the number of b (that is, the number of solutions) is given by $1+\left(\dfrac{D(a, c)}{p}\right)$, so we get

$$\begin{aligned}(C(v_3)) &= \sum_{a,c=0}^{p-1}\left(1+\left(\frac{D(a, c)}{p}\right)\right) \\ &= p^2+\sum_{a,c=0}^{p-1}\left(\frac{D(a, c)}{p}\right),\end{aligned}$$

where $\left(\frac{*}{p}\right)$ is the Legendre symbol. For a fixed a such that $az'+x'\not\equiv 0 \bmod p$, $D(a, c)$ runs through Z/pZ, and the partial sum of these elements in the second term is zero. For the unique solution a of $az'+x'\equiv 0 \bmod p$, we get $D(a, c)\equiv T_3 \bmod p$. Thus, we get

$$(C(v_3))=p^2+p\left(\frac{T_3}{p}\right)$$

and $(\mathrm{R}\,1)=(\mathrm{L}\,1)$.

Subcase (b). Assume that $z'\in p\mathbb{Z}_p$ and $x\in \mathbb{Z}_p^\times$. Then, $v_1\in \mathbb{Z}_p^5$ and $v_2\notin \mathbb{Z}_p^5$ (for any parameter). We get

$$C(v_2')=\{(a, q);\ 0\leqq a\leqq p-1,\ q^2x'+2qt'-w'\equiv 0 \bmod p\}.$$

The discriminant of $q^2x'+2qt'-w'$ with respect to q is $t'^2+x'w'\equiv T_3 \bmod p$. So, $\#(C(v_2'))=p\left(1+\left(\frac{T_3}{p}\right)\right)$. We get $\#(C(v_3))=p^2$, and thus $(\mathrm{R}\,1)=(\mathrm{L}\,1)$.

Subcase (c). Assume that $x', z'\in p\mathbb{Z}_p$ and $t'\in \mathbb{Z}_p^\times$. Then, $v_1\in M$,

$$\#(C(v_2))=\#(C(v_2'))=p \quad\text{and}\quad \#(C(v_3))=p^2.$$

In this case, $T_3\equiv t'^2 \bmod p$, so $\left(\frac{T_3}{p}\right)=1$, and we get $(\mathrm{R}\,1)=(\mathrm{L}\,1)$.

Subcase (d). Assume that $x', z', t'\in p\mathbb{Z}_p$ and $w'\in \mathbb{Z}_p^\times$. Then, $v_1\in M$, $v_2'\notin M$, $\#(C(v_2))=p$, and $\#(C(v_3))=p^2$. In this case, $T_3\equiv 0 \bmod p$.

Subcase (e). Assume that $x', z', t', w'\in p\mathbb{Z}_p$ and $y'\in \mathbb{Z}_p^\times$. Then, $v_1\in M$, $v_3\notin M$, $\#(C(v_2))=p$, $\#(C(v_2'))=p^2$, and $T_3\equiv 0 \bmod p$.

Thus, (3. 7) is proved in the Case (II). The equation (3. 9) is proved similarly: (R 2) is obtained by calculating in each case (a) ... (e) as above, and we get

$$(\mathrm{R}\,2)=(\mathrm{L}\,2)=(p^3+p^2)\left(1+\frac{T_3}{p}\right)\Bigg).$$

We omit the details here.

Case (III). We prove (3. 7). We have $\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} UY\notin L$ for any $U\in R(p)$, and $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} Y\in L$ only for $x=0$ (if $0\leqq x\leqq p-1$). So, we get $(\mathrm{L}\,1)=p^{\frac{1}{2}}$. Now, we calculate $(\mathrm{R}\,1)$.

Subcase (a). Assume that $t'\in p^{-1}\mathbb{Z}_p^\times$. Then, x' or $z'\in p^{-1}\mathbb{Z}_p^\times$, because we have assumed $T_3\in \mathbb{Z}_p$. If $z'\in \mathbb{Z}_p$ and $x'\in p^{-1}\mathbb{Z}_p^\times$, then, $v_1, v_2, v_3\notin M$ and

$$C(v_2')=\{(a, q);\ y'-ax'\equiv t'+qx'\equiv -q^2x'-az'+w'-2qt'\equiv 0 \bmod p\}.$$

The first two conditions in $C(v_2')$ imply $a \equiv \frac{y'}{x'} \bmod p$ and $q \equiv -\frac{t'}{x'} \bmod p$. Now,

$$\begin{aligned} -q^2x'-az'+w'-2qt' &\equiv -q(qx'+t')-qt'-az'+w' \bmod p \\ &\equiv t'x'^{-1}(qx'+t')-qt'-y'z'x'^{-1}+w' \bmod p \\ &\equiv -\frac{T_3}{x'} \equiv 0 \bmod p. \end{aligned}$$

Thus, we get $(\mathrm{R}\,1)=p^{\frac{1}{2}}$. If $z' \in p^{-1}\mathbb{Z}_p^\times$, then, $v_1, v_2, v_2' \notin M$, and

$$C(v_3)=\left\{(a,b,c);\ a \equiv -\frac{x'}{z'} \bmod p,\ c \equiv \frac{w'}{z'} \bmod p,\ b \equiv \frac{t'}{z'} \bmod p, \ \text{and} \right.$$
$$\left. y'-cx'-2bt'+(b^2-ac)z'+aw' \equiv 0 \bmod p \right\}.$$

If the first three conditions in $C(v_3)$ are satisfied, the last condition is automatic, because

$$y'-c(x'+az')-2bt'+b^2z'+aw' \equiv \frac{T_3}{z'}+\frac{t'-bz'}{z'} \equiv 0 \bmod p.$$

Thus, we get $(\mathrm{R}\,1)=p^{\frac{1}{2}}=(\mathrm{L}\,1)$.

Subcase (b). Assume that $t' \in \mathbb{Z}_p$ and $z' \in p^{-1}\mathbb{Z}_p^\times$. Then, $v_1, v_2, v_2' \notin M$ and

$$C(v_3)=\{(a,b,c);\ b=0,\ x'+az',\ w'-cz' \in \mathbb{Z}_p\}.$$

So, $\#(C(v_3))=1$ and $(\mathrm{R}\,1)=(\mathrm{L}\,1)$.

Subcase (c). Assume that $t', z' \in \mathbb{Z}_p$, $x' \in p^{-1}\mathbb{Z}_p^\times$. Then, $v_1, v_2, v_3 \notin M$ and

$$C(v_2')=\left\{(a,q);\ q=0,\ a \equiv \frac{y'}{x'} \bmod p,\ w' \equiv az' \bmod p\right\}.$$

But, $w'-az' \equiv \frac{T_3}{x'} \equiv 0 \bmod p$, so $\#(C(v_2'))=1$.

Subcase (d). Assume that $t', z', x' \in \mathbb{Z}_p$ and $w' \in p^{-1}\mathbb{Z}_p^\times$. Then, $v_1, v_2', v_3 \notin M$ and $\#(C(v_2))=1$.

Subcase (e). Assume that $t', z', x', w' \in \mathbb{Z}_p$, $y' \in p^{-1}\mathbb{Z}_p^\times$. Then, $z' \in p\mathbb{Z}_p$, because $T_3 \in \mathbb{Z}_p$. So, $v_1 \in M$ and $v_2, v_2', v_3 \notin M$. Thus, (3. 7) is proved in the Case (III). (3. 9) is similarly proved and $(\mathrm{R}\,2)=(\mathrm{L}\,2)=p^2+p$. We omit the details here.

Case (IV). The proof for this case is quite long. We give some hints for the proof and omit the details here. Y_1 satisfies one of the following conditions

(a) $z \in \mathbb{Z}_p^\times$,

(b) $z \in p\mathbb{Z}_p$, $x \in \mathbb{Z}_p^\times$,

(c) $z, x \in p\mathbb{Z}_p$, $t \in \mathbb{Z}_p^\times$,

(d) $x, z, t \in p\mathbb{Z}_p$, $w \in \mathbb{Z}_p^\times$,

(e) $x, z, t, w \in p\mathbb{Z}_p$, $y \in \mathbb{Z}_p^\times$,

as in the proof of the case (II). Y_2 also satisfies one of the same conditions

(a′) $z' \in \mathbb{Z}_p^\times, \dots$ etc.

The proof is divided into subcases, where Y_1, Y_2 satisfy the conditions ((a), (a')), ((b), (b')),..., or ((e), (e')). The proof in the case ((a), (a')) is most complicated. Here, we sketch the proof of (3. 7) in this case. It is obvious that

$$(\mathrm{L}\,1)=1+p+\begin{cases} p^{\frac{3}{2}}\left(\dfrac{T_1}{p}\right), & \text{if } p\mid T_3,\ p\mid D,\\ p^{\frac{3}{2}}\left(\dfrac{T_3}{p}\right), & \text{if } p\nmid T_3,\ p\mid D,\\ 0, & \text{if } p\nmid D,\end{cases}$$

$$+\begin{cases} p^2, & \text{if } zt'-z't\equiv zx'-xz'\equiv zy'-yz'\equiv zw'-z'w\equiv 0 \bmod p,\\ 0, & \text{otherwise.}\end{cases}$$

We get $v_1\notin M$ always, and $v_2\in M$, if and only if $zx'-z'x\equiv 0 \bmod p$. $\#(C(v_2'))$ and $\#(C(v_3))$ are given as follows:

(i) If $zx'-xz'\notin p\mathbb{Z}_p$, then

$$\#(C(v_2'))=1+\left(\frac{A}{p}\right),$$

and

$$\#(C(v_3))=\begin{cases} p+(p-1)\left(\dfrac{A}{p}\right), & \text{if } p\mid D \text{ and } A\notin p\mathbb{Z}_p,\\ p+p\left(\dfrac{B}{p}\right), & \text{if } p\mid D \text{ and } A\in p\mathbb{Z}_p,\\ p-\left(\dfrac{B}{p}\right), & \text{if } p\nmid D,\end{cases}$$

where $A=z^2T_1-zz'T_2+z'^2T_3$ and $B=x^2T_1-xx'T_2+x'^2T_3$.

(ii) If $zx'-xz'\in p\mathbb{Z}_p$ and $zt'-tz'\notin p\mathbb{Z}_p$, then

$$\#(C(v_2'))=1$$

and

$$\#(C(v_3))=\begin{cases} 2p-1, & \text{if } p\mid D,\\ p-1, & \text{if } p\nmid D.\end{cases}$$

(iii) If $zx'-xz'$, $zt'-tz'\in p\mathbb{Z}_p$ and $zw'-z'w\notin p\mathbb{Z}_p$, then

$$(C(v_2'))=0$$

and

$$(C(v_3))=\begin{cases} p+p\left(\dfrac{T_3}{p}\right), & \text{if } p\mid D,\\ p, & \text{if } p\mid D.\end{cases}$$

(iv) If $zx'-xz'$, $zt'-tz'$, $zw'-wz'\in p\mathbb{Z}_p$ and $zy'-yz'\notin p\mathbb{Z}_p$, then

$$\#(C(v_2'))=p,$$

$$\#(C(v_3))=0.$$

(v) If $zx'-xz', zt'-tz', zw'-z'w, zy'-yz' \in p\mathbb{Z}_p$, then

$$\#(C(v_2'))=p$$

and

$$\#(C(v_3))=p^2+p\left(\frac{T_3}{p}\right).$$

In the case (i), if $p|D$, $p\nmid T_3$, and $p\nmid A$, then

$$A \equiv T_3(z'-2^{-1}T_3^{-1}zT_2)^2 \bmod p \quad \text{and} \quad \left(\frac{A}{p}\right)=\left(\frac{T_3}{p}\right).$$

If $p|D$, $p\nmid T_3$, and $p|A$, then $T_2 \equiv 2zz'T_3 \bmod p$, $T_1 \equiv z^{-2}z'^2T_3 \bmod p$, and

$$B \equiv (xz'-zx')^2\, T_3 \bmod p,$$

so $\left(\frac{B}{p}\right)=\left(\frac{T_3}{p}\right)$. If $p|D$, $p|T_3$, then $p|T_2$ and $\left(\frac{A}{p}\right)=\left(\frac{T_1}{p}\right)$, and $p|A$ implies $p|T_1$ and $p|B$.

In the case (ii), if $p|D$ and $p|T_3$, then $z^2T_1 \equiv (zt'-tz')^2 \bmod p$ and $\left(\frac{T_1}{p}\right)$. If $p|D$ and $p\nmid T_3$, then $(zt'-t'z)^2 \equiv z^2T_1-zz'T_2+z'^2T_3 \equiv T_3(z'-2^{-1}T_3^{-1}zT_2)^2 \bmod p$, and $\left(\frac{T_3}{p}\right)=1$.

In the case (iii), we have $T_1 \equiv z^{-1}z'T_2-z^{-2}z'^2T_3$, and if $p|D$ and $p|T_3$, then $T_1 \equiv 0 \bmod p$.

In the case (iv), we have $D \equiv -(yz'-zy')^2 \bmod p$, so $p\nmid D$.

In the case (v), we have $T_1 \equiv z^{-2}z'^2T_3 \bmod p$ and $T_2 \equiv 2z'z^{-1}T_3 \bmod p$, so $p|D$ and $\left(\frac{T_1}{p}\right)=\left(\frac{T_3}{p}\right)$.

Combining these data, we get (R 1) = (L 1) in the case ((a), (a')).

We omit the proof of the other cases.

Case (V) and (VI). The proofs of these cases are also long, but more or less similar to the above proofs. We omit them here.

Thus, Theorem 3. 6 and 3. 8 are proved.

§ 4. Main Theorem

In this section, we prove our Main Theorem, and define *L*-functions.

Main Theorem. *Let κ be the representation of $Sp(2)$ which corresponds to a Young diagram (f_1, f_2) with $f_1 \geqq f_2 \geqq 0$, $f_1+f_2=$ even. For $\varphi \in M_\kappa(U)$, define*

$$\sigma(\varphi)=\Phi \in S\left(\det^{\frac{f_1-f_2+5}{2}} \otimes \mathrm{Sym}(f_2), \Gamma_0'(d')\right)$$

as in § 2. 3. *Assume that $T(1,1,p,p)\,\varphi=\lambda(p)\,\varphi$ and*

$$T(1,p,p^2,p)\,\varphi=\omega(p)\,\varphi \qquad (\lambda(p), \omega(p) \in \mathbb{C}).$$

Then, we have

$$\varepsilon_p T_0(1, p, p^2, p)\, \Phi = \lambda(p)\, \Phi, \quad \text{and} \quad T_0(p, p, p^3, p^3)\, \Phi = \omega(p)\, \Phi,$$

where $\varepsilon_p = 1$ *or* -1 *according as* $p \equiv 1$ *or* $3 \bmod 4$, *respectively.*

Proof. It is clear that the right hand sides of (3. 7) and (3. 9) do not depend on the choice of $\{h_s\}$. It is well known that we can take $\{h_s\}$ so that

$$U_p h_0 U_p = \coprod_{s=1}^{v} U_p h_s = \coprod_{s=1}^{v} h_s U_p.$$

Now, for $\bar{g}_i$ such that $K_p \omega K_p = \coprod_{i=1}^{u} K_p \bar{g}_i$, assume that

$$\sum_{i=1}^{u} \pi(\bar{g}_i^{-1}) f_p = c \sum_{s=1}^{v} f_p(y\rho(h_s)^{-1})$$

for some constant c. Then,

$$\begin{aligned}
\sum_{i=1}^{u} \Phi(\bar{g}\bar{g}_i) &= \int_{R^\times G'_Q \backslash G'_A} \left\langle \sum_{i=1}^{u} \sum_{y \in X} \pi(\bar{g}\bar{g}_i^{-1}) f(y\rho(h)), \varphi(h) \right\rangle dh \\
&= \int_{R^\times G'_Q \backslash G'_A} \left\langle \sum_{y \in X} \pi(\bar{g}) \left(\sum_{i=1}^{u} \pi(\bar{g}_i^{-1}) f \right) (y\rho(h)), \varphi(h) \right\rangle dh \\
&= c \sum_{s=1}^{v} \int_{R^\times G'_Q \backslash G'_A} \left\langle \sum_{y \in X} (\pi(\bar{g}) f)\, (y\rho(hh_s^{-1})), \varphi(h) \right\rangle dh \\
&= c \int_{R^\times G'_Q \backslash G'_A} \left\langle \sum_{y \in X} (\pi(\bar{g}) f)\, (y), \sum_{s=1}^{v} \varphi(hh_s) \right\rangle dh.
\end{aligned}$$

Thus, taking the normalizing constant defined in (3. 3) and (3. 4) into account, and putting $c = p^{\frac{1}{2}}$, or p, as in Theorem 3. 6 and 3. 8, we get $T_0(1, p, p^2, p)\, \Phi = \lambda(p)\, \Phi$ and

$$T_0(p, p, p^3, p^3)\, \Phi = T_0(p, p, p, p)\, T_0(1, 1, p^2, p^2)\, \Phi = \omega(p)\, \Phi.$$

q.e.d.

In general, take a common eigen-form $\Phi \in S(\Gamma_0(d'), \det^{\frac{k}{2}} \otimes \mathrm{Sym}(k'))$ of all the Hecke operators $T_0(1, p, p^2, p)$ and $T(p, p, p^3, p^3)$ $(p \nmid d')$, and denote by $\lambda(p)$ and $\omega(p)$ the eigen values. We define L-function of Φ by:

$$L(s, \Phi) =$$

$$\prod_p \left(1 - \lambda(p)\, p^{-s} + \left(p\omega(p) + p^{2k'+k-4}(1+p^2)\right) p^{-2s} - \lambda(p)\, p^{2k'+k-2-3s} + p^{4k'+2k-4-4s}\right)^{-1},$$

where p runs through all primes which do not divide d'.

On the other hand, put $\bigcup_g UgU = \coprod_{s=1}^{v} Uh_s$ (disjoint), where g runs through elements of G_A with the multiplicator $n(g)=n$. Let $T(n)$ be the usual Hecke operator acting on $M_\kappa(U)$:

$$T(n)\,\varphi = n^{\frac{f_1+f_2}{2}} \sum_{s=1}^{v} \kappa(h_s)\,\varphi(hh_s),$$

where $\varphi \in M_\kappa(U)$. Let φ be a common eigen form of all $T(n)$: $T(n)\,\varphi = \mu(n)\,\varphi$. Then, it is standard to define $L(s, \varphi)$ by the "denominator" of $\sum_{n=1}^{\infty} \mu(n)\,n^{-s}$, that is, it is given by:

$$\prod_{p \nmid d} \left(1 - \mu(p)\,p^{-s} + \left(\mu(p)^2 - \mu(p^2) - p^{f_1+f_2+2}\right)p^{-2s} - \mu(p)\,p^{f_1+f_2+2-3s} + p^{2f_1+2f_2+6-4s}\right),$$

up to finitely many bad Euler factors.

Corollary. *Notations and assumptions being the same as above, we get*

$$L(s, \varphi) = L\big(s, \sigma(\varphi)\big)$$

up to finitely many bad Euler factors.

The meaning of $L(s, \sigma(\varphi))$ is explained as follows: Let $f(Z)$ be the "classical" automorphic form which corresponds to $\sigma(\varphi)$ as in (2. 2). Let

$$f(Z) = \sum_{T>0} a(T) \exp\big(2\pi i \operatorname{tr}(TZ)\big)$$

be the Fourier expansion, where T runs through positive symmetric half integral symmetric matrices, and $a(T)$ are vectors in $\mathbb{C}^{f_2+1}$.

Proposition 4. 1. *For any prime $p \nmid d'$ and fixed $N \in M_2(\mathbb{Q})$, we have*

$$\sum_{n=0}^{\infty} \sum_{\substack{\det M = p^n \\ M \in SL_2(\mathbb{Z}) \backslash M_2(\mathbb{Z})}} \tau(M)^{-1}\, a(MN{}^tM)\, t^n$$

$$= \frac{a(N) + a_1(N)\,t + a_2(N)\,t^2 + a_3(N)\,t^3}{1 - \lambda(p)\,t + \big(p\omega(p) + p^{2k'+k-4}(p^2+1)\big)\,t^2 - \lambda(p)\,p^{2k'+k-2}\,t^3 + p^{4k'+2k-4}\,t^4}$$

as vectors in $\mathbb{C}^{f_2+1}$, where t is variable, $\tau = \operatorname{Sym}(f_2)$, and $a_i(N)$ is determined automatically from the above relation.

Proof. This has been proved for $\mathbb{C}$-valued Siegel modular forms of half-integral weight by Juravlev [15]. (His results include the case of general degree.) The present author also obtained independently this Proposition for degree two, including vector valued forms. The proof will be omitted here.

§ 5. Examples

In this section, we give some examples of $\sigma(\varphi)$. We assume in this section that $d=2$. We take representations κ_i $(i=1, 2, 3)$ which correspond with the Young diagram $(0, 0)$, $(2, 2)$, or $(8, 0)$, respectively. Then, $\dim M_{\kappa_i}(U)=1$ for $i=1, 2, 3$ (cf. [7]). In these cases, $\sigma(\varphi)$ is given (up to constant) respectively as follows:

$$F_i(Z)=\sum_{\substack{x,y\in \mathfrak{O}\\ t,s\in \mathfrak{O}}} f_i(x, y, t, s)\exp\left(2\pi i \operatorname{tr}\left(Z\begin{pmatrix} n(x)+t^2 & ts+\dfrac{\operatorname{tr}(x\bar{y})}{2} \\ ts+\dfrac{\operatorname{tr}(x\bar{y})}{2} & n(y)+s^2\end{pmatrix}\right)\right)$$

for $i=1, 2, 3$, where $Z\in\mathfrak{H}_2$, and

$$f_1=1,$$

$$f_2=\begin{pmatrix} 4t^2-n(x) \\ 4ts-\dfrac{\operatorname{tr}(x\bar{y})}{2} \\ 4s^2-n(y)\end{pmatrix},$$

$$f_3=g(x, y)+g(\bar{x}, \bar{y}).$$

Here, $g(x, y)=(x\bar{y})_2^4+(x\bar{y})_3^4+(x\bar{y})_4^4-3\left((x\bar{y})_2^2(x\bar{y})_3^2+(x\bar{y})_3^2(x\bar{y})_4^2+(x\bar{y})_4^2(x\bar{y})_2^2\right)$, where, for any $x\in\mathbb{H}$, we define x_i by $x=x_1+x_2 i+x_3 j+x_4 k$ $(x_i\in\mathbb{R})$, when $i^2=j^2=-1$, $ij=-ji=k$.

These F_i do not vanish identically. F_1 and F_2 are not cusp forms, and L-functions are given respectively as follows:

$$L(F_1, s)=\zeta(s)\,\zeta(s-1)\,\zeta(s-2)\,\zeta(s-3),$$

$$L(F_2, s)=\zeta(s)\,\zeta(s-7)\,L(s, \Delta_8),$$

up to Euler two-factors, where Δ_8 is the unique cusp form of weight 8 of $\mathfrak{H}_1$ belonging to $\Gamma_0(2)$. The above assertion for F_1 is obvious, and the assertion for F_2 is a consequence of Ihara's result in [13]. The form F_3 is a cusp form by virtue of Andrianov and Maloletkin [2].

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