On the Immersion of an Algebraic Ring into a Field.

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A set M with an operation of composition is called a semigroup if this operation satisfies the following conditions[1]):

a) To every two elements a and b of M corresponds another element c of M, the product of a and b, i. e.

$$c = ab.$$

b) The operation of composition is associative, i. e. whatever be three elements a, b, c of M we always have

$$a(bc) = (ab)c.$$

c) Both divisions are univoque, i. e. if

$$ax = ay \quad \text{or} \quad xb = yb,$$

then

$$x = y.$$

It can be easily proved that every commutative semigroup can be „immersed" (eingebettet) into a group[2]). However, the analogous question concerning non-commutative semigroups, as far as we know, remained unsolved.

Prof. A. Suschkewitsch has published a proof[3]) that every semigroup can be immersed into a group. However, we shall construct (in § 2 of the present paper) a semigroup which can not be immersed into a group; thus, Professor Suschkewitsch's result fails to be true.

An analogous problem exists for rings, viz. can every ring without divisors of zero (Nullteilern) be immersed into a field[4])?

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[1]) See e. g. O. Schmidt. The Abstract Theory of Groups (in Russian). Kiev 1916, p. 58.

[2]) This can be proved in the same way as the theorem is proved that every commutative ring without divisors of 0 can be immersed into a field $\left(\text{i. e. considering „quotients"} \frac{a}{b}\right)$. See B. L. v. d. Waerden. Moderne Algebra. Berlin 1930, Bd. I, S. 47—48.

[3]) A. Suschkewitsch, Über die Erweiterung der Semigruppe bis zur ganzen Gruppe. Commun. Soc. Math. Kharkoff et Inst. Sci. de Math. et Mécan. Univ. Kharkoff (4) 12 (1935), 81—86 (in Russian).

[4]) See e. g. v. d. Waerden, op. cit. S. 49.

It is well known that in the case of commutative rings such immersion is always possible. We shall show that in the general case (of non-commutative rings) it is not always so; viz. we shall construct (in § 3 of this paper) a ring without divisors of zero, which cannot be immersed into an algebraic field. In this way a problem mentioned by van der Waerden finds its solution[2]).

We also have found the necessary and sufficient conditions for the possibility of immersion of a semigroup into a group. However these are too complicated to be included in this paper.

§ 1.

The condition Z.

The following condition is necessary for the possibility of immersion of a semigroup $\mathfrak{H}$ into a group.

Condition Z. Whatever be eight elements A, B, C, D, X, Y, U, V of $\mathfrak{H}$ such that

$$AX = BY$$
$$CX = DY$$
$$AU = BV$$

we always have

$$CU = DV.$$

Proof. We have

$$B^{-1}A = YX^{-1}$$
$$D^{-1}C = YX^{-1}$$
$$B^{-1}A = VU^{-1}$$

whence $D^{-1}C = VU^{-1}$, or $CU = DV$ q. e. d.

Hence follows that if a semigroup $\mathfrak{H}$ does not satisfy the condition Z then this semigroup can not be immersed into a group. In the next § we shall construct a semigroup not satisfying the condition Z.

§ 2.

The construction of the „non-immersible“ semigroup $\mathfrak{H}$.

Consider all possible finite sequences of eight letters a, b, c, d, x, y, u, v. Such sequences we shall call „words“. The number of letters contained in a word shall be called its „length“.

The words

$$\text{(A)} \qquad \begin{cases} ax \text{ and } by \\ cx \text{ and } dy \\ au \text{ and } bv \end{cases}$$

we shall call „corresponding“.

Consider now a word α. It may contain one of the combinations ax, by, cx, dy, au, bv (i. e. be of the form $\ldots mn \ldots$, where mn is one of the enumerated combinations of letters). If we replace this pair of letters by the corresponding pair taken from the table (A) we shall obtain another word β. We shall say that β is obtained from α by an elementary transformation. We shall call equivalent with α a word γ which can be obtained from the word α by a finite number of elementary transformations. We shall write in that case $\alpha \sim \gamma$. The following properties of the equivalence are evident:

1. $\alpha \sim \alpha$,
2. If $\alpha \sim \beta$, then $\beta \sim \alpha$,
3. If $\alpha \sim \beta$ and $\beta \sim \gamma$ then $\alpha \sim \gamma$,
4. Two equivalent words have the same length.

By the „product" of two words α and β we shall understand the word $\alpha\beta$ obtained by writing down first the word α and after it the word β. We have:

5. If $\alpha \sim \beta$ and $\gamma \sim \delta$ then $\alpha\gamma \sim \beta\delta$.

Consider now three consecutive letters mnp of a word $[\ldots mnp \ldots]$. Then, if the pair mn admits an elementary transformation then such transformation is impossible for the pair np. And conversely: if we can make an elementary transformation on np then this is impossible on mn. In fact we see at once from the table (A) that if np admits an elementary transformation then n must be one of the letters a, b, c, d, while if such transformation is possible on mn then n must be one of the letters x, y, u, v. Consequently these two cases are mutually exclusive.

Hence we can easily deduce the following two properties of the equivalence:

6. If $\alpha\beta \sim \alpha\gamma$ or $\beta\delta \sim \gamma\delta$ then $\beta \sim \gamma$.
7. If $\alpha\beta \sim \gamma\delta$ and the words α and γ have the same length then

$$\alpha = \mu\, m, \quad \beta = n\nu,$$
$$\gamma \sim \mu\, m', \quad \delta \sim n'\,\nu,$$
$$m\,n \sim m'\,n',$$

where m, n, m', n' are letters (i. e. each of the letters m, n, m', n' denotes one of the eight letters a, b, c, d, x, y, u, v). If α or β has its length equal to 1 then the factor μ, resp. ν, disappears.

The properties 1, 2, 3 together show that all words can be divided into classes (Äquivalenz-Klassen) of mutually equivalent words. These classes will be elements of our semigroup $\mathfrak{H}$. The product AB of two classes A and B shall be defined as the class C containing a word $\alpha\beta$ where α belongs to A and β belongs to B. The property (5) shows that this class C will not depend on the choice of the words α and β in the classes A and B resp.. From the property (6) follows that if

$$AB = AC \text{ or } BD = CD$$

then

$$B = C.$$

Therefore we see that these classes constitute a semigroup $\mathfrak{H}$. In what follows we shall call (for the sake of convenience) (α) the class containing the word α. Then the definition of the classes and of the composition of classes can be written down as follows:

8. $(\alpha) = (\beta)$ if and only if $\alpha \sim \beta$.
9. $(\alpha)(\beta) = (\alpha\beta)$.

By the length of a class A we shall understand the length of a word α belonging to A. By the property 4. this length does not depend on the choice of α in A.

We shall prove now that the semigroup $\mathfrak{H}$ can not be immersed into a group. To this end it is sufficient to prove that $\mathfrak{H}$ does not satisfy the condition Z. In fact we have

$$(a)(x) = (b)(y)$$
$$(c)(x) = (d)(y)$$
$$(a)(u) = (b)(v).$$

Nevertheless (in contradiction with Z) $(c)(u) \neq (d)(v)$ because no elementary transformations can be effected on the word cu and therefore this word is equivalent to no other word except itself.

We have proved thus that the semigroup $\mathfrak{H}$ does not satisfy the condition Z, and therefore it cannot be immersed into a group, q. e. d..

§ 3.

The construction of a non-immersible ring.

Consider the ring $\mathfrak{R}$ of all linear forms

$$\Sigma\, k_i X_i$$

where X_i are the elements of the semigroup $\mathfrak{H}$ constructed in the preceding § and k_i are rational numbers only a finite number of which is different from zero.

The sum and product of two such forms are defined according to the ordinary rules of addition and multiplication[5]), i. e.

$$\Sigma\, k_i X_i + \Sigma\, l_i X_i = \Sigma\,(k_i + l_i)\, X_i$$
$$\Sigma\, k_i X_i \cdot \Sigma\, l_i X_i = \sum_{(i,j)} k_i l_j X_i X_j;$$

we shall prove that this ring $\mathfrak{R}$ has no divisors of 0 (Nullteiler).

[5]) $\mathfrak{R}$ can be defined as a hypercomplex system of an infinite rank with the semigroup $\mathfrak{H}$ as basis. See e. g. v. d. Waerden, Bd. II, S. 149.

To this end we shall prove first the following property of the semigroup $\mathfrak{H}$.

Lemma. Let $X_1, X_2, X_3, Y_1, Y_2, Y_3$ be elements of $\mathfrak{H}$. If

$$X_1 Y_1 = X_2 Y_2$$
$$X_1 Y_2 = X_3 Y_3$$

and if the elements X_1, X_2, X_3 have the same length then either $Y_1 = Y_2$ or $Y_2 = Y_3$.

Proof. The relation

$$X_1 Y_1 = X_2 Y_2$$

implies, by the property 7.,

$$X_1 = X' \cdot (m) \qquad Y_1 = (n) \cdot Y'$$
$$X_2 = X' \cdot (m') \qquad Y_2 = (n') \cdot Y'$$
$$(mn) = (m'n')$$

where one or both of the factors X', Y' can disappear. In the same way the relation

$$X_1 Y_2 = X_3 Y_3$$

implies

$$X_1 = X' \cdot (m) \qquad Y_2 = (n') \cdot Y'$$
$$X_3 = X' \cdot (p) \qquad Y_3 = (q) \cdot Y'$$
$$(mn') = (pq).$$

Here m, n, m', n', p, q, denote words of length 1. If now $Y_1 \neq Y_2$ and $Y_2 \neq Y_3$, then $n \neq n'$ and $n' \neq q'$, while

$$(m'n') = (mn)$$
$$(mn') = (pq).$$

We shall prove that this is impossible.

In fact the inspection of the table (A) will show us that n' must be either x or y because no other letter is found twice as a second component in this table. In the same way m can be only a or b. Therefore mn' must be ax or by. Suppose for instance that $mn' = ax$, then $mn = au$ and $m'n' = bv$. We thus arrive at a contradiction because n' had to be x or y. The supposition $mn' = by$ will similarly lead us to a contradiction. Thus either $n = n'$ or $n' = p$. In the first case we have $Y_1 = Y_2$, in the second $Y_2 = Y_3$, q. e. d..

We can easily prove now that the ring $\mathfrak{R}$ has no divisors of 0. In fact let

$$\text{(B)} \qquad \Sigma k_i X_i \cdot \Sigma l_j Y_j = \sum_{(i,j)} k_i l_j X_i Y_j = 0.$$

The longest terms of the sum $\sum_{(i,j)} k_i l_j X_i Y_j$ have the sum 0 because they cannot vanish in combination with the shorter terms. But these longest terms are obtained by multiplying the longest terms of $\Sigma k_i X_i$ with the longest terms of $\Sigma l_i Y_j$. Therefore if we remove from $\Sigma k_i X_i$ and $\Sigma l_j Y_j$ all but their longest terms the relation (B) remains true. From what was just said it is evident that we can suppose that all X_i's (as well as all Y_i's) are of the same length.

So let

$$\Sigma k_i X_i \cdot \Sigma l_j Y_j = \sum_{(i,j)} k_i l_j X_i Y_j = 0$$

and let all X_i's be different and of the same length; in the same way let all Y_j's be different and of the same length; besides let $k_1 l_1 \neq 0$.

In order that $k_1 l_1 X_1 Y_1 (\neq 0)$ should combine with one or more terms to give 0 it is necessary that there exist such i and j $(i \neq 1,\ j \neq 1)$ that $k_i \neq 0$, $l_j \neq 0$ and that

$$X_1 Y_1 = X_i Y_j.$$

But then the term $k_1 l_j X_1 Y_j$ is different from zero; in order that this term could vanish in combination with other terms of the sum it is necessary that there exist such i' and j' $(j \neq j')$ that

$$X_1 Y_j = X_{i'} Y_{j'}.$$

But then by our lemma either $Y_1 = Y_j$ or $Y_j = Y_{j'}$ which is impossible.

Thus the ring $\mathfrak{R}$ has no divisors of 0. Nevertheless $\mathfrak{R}$ cannot be immersed into a field because $\mathfrak{H}$ would then be immersed into a group (viz. the multiplicative group of the field), which is impossible as we have seen.

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