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SOME NEW CRITERIONS FOR SEQUENCES WHICH SATISFY DUFFIN—SCHAEFFER CONJECTURE, I.

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In [4, p. 255], Duffin and Schaeffer gave following conjecture (abbreviated D.S.C.):

Let $\{q_i\}$ be a one-to-one sequence of positive integers and $\{f(q_i)\}$ be a sequence of positive real numbers. If the series

$$\Sigma\, \varphi(q_i) f(q_i)$$

is divergent (φ stands for Euler totient function), then for almost all u the diophantine inequality

$$\left| u - \frac{p}{q_i} \right| < f(q_i)$$

has for infinitely many i an integer solution p with g.c.d. $(p, q_i) = 1$.

In [1] we proved following criterion for D.S.C., see [1, Theorem 5 and Proof of Theorem 14]:

Let $N(q_i, q_j, t)$ denote the number of pairs of positive integers x, y for which

$$0 < \frac{x}{q_i} - \frac{y}{q_j} \leqq t$$

$$0 < x < q_i, \quad 0 < y < q_j, \quad (x, q_i) = (y, q_j) = 1 \tag{1}$$

Theorem 1. Let there for infinitely many positive integer m exist positive constants c_m, c'_m with $c'_m \to 0$ for $m \to \infty$, such that

$$\sum_{m < i,\, j \leqq n} N(q_i, q_j, t) \leqq c_m t \Big(\sum_{m < i \leqq n} \varphi(q_i) \Big)^2 + c'_m \Big(\sum_{m < i \leqq n} \varphi(q_i) \Big) \tag{2}$$

for all sufficiently large n and for every $t > 0$. Then $\{q_i\}$ satisfies D.S.C. with every nonincreasing $\{f(q_i)\}$.

Using Theorem 1 and a new estimation for the function

$$\varphi_q(x) = \sum_{\substack{i \leqq x \\ (i,\, q) = 1}} 1 \tag{3}$$

we can find new sequences $\{q_i\}$ which satisfy D.S.C. Put

$$d_{ij}=(q_i,\ q_j),\quad q_{ij}=q_iq_j/d_{ij}^2$$

Theorem 2. If

$$\sum \frac{\log^2 q_{ij}}{q_{ij}} \frac{\varphi(q_i)}{q_i} \frac{\varphi(q_j)}{q_j} < +\infty \tag{4}$$

then $\{q_i\}$ satisfies D.S.C. for every $\{f(q_i)\}$.

Proof. By [1, Lemma 8] a number of pairs of positive integers x, y which satisfies (1) and

$$\frac{x}{q_i}-\frac{y}{q_j}=\frac{a}{d_{ij}q_{ij}}$$

is not greater than

$$\varphi(d_{ij})\frac{\varphi(d_{ij}^o)}{d_{ij}^o}\frac{(a,\ d_{ij}^o)}{\varphi((a,\ d_{ij}^o))}$$

where $(a,\ q_{ij})=1$ and

$$d_{ij}^o=\prod_{\substack{p\mid d_{ij}\\ p\nmid q_{ij}}} p$$

From it

$$N(q_i,\ q_j,\ t)\leqq \varphi(d_{ij})\frac{\varphi(d_{ij}^o)}{d_{ij}^o}\sum_{\substack{a\leqq td_{ij}q_{ij}\\ (a,\ q_{ij})=1}}\frac{(a,\ d_{ij}^o)}{\varphi((a,\ d_{ij}^o))}=$$

$$=\varphi(d_{ij})\frac{\varphi(d_{ij}^o)}{d_{ij}^o}\sum_{d\mid d_{ij}^o}\frac{|\mu(d)|}{\varphi(d)}\sum_{\substack{a\leqq td_{ij}q_{ij}/d\\ (a,\ q_{ij})=1}}1 \tag{5}$$

Further we have a need of a good estimation of the function $\varphi_q(x)$. Denote

$$\varphi_q(x)=\prod_{\substack{p>x\\ p\mid q}} p$$

where p are prime numbers.

Lemma 1.

$$\varphi_q(x)\leqq c_0 x\frac{\varphi(q)}{q}\frac{q(x)}{\varphi(q(x))} \tag{6}$$

for every positive x and integer q. If $x\geqq\log q$ then

$$\varphi_q(x) \leqq c_0 x \frac{\varphi(q)}{q} \tag{7}$$

If $2 \leqq x \leqq \log q$ and $q \geqq 3$ then

$$\varphi_q(x) \leqq c_0 x \frac{\varphi(q)}{q} \frac{\log\log q}{\log x} \tag{8}$$

Proof. Let $x \geqq p$ for every prime $p \backslash q$. Then, see [2, p. 105], (7) is true. Always, if $i \leqq x$ then

$$\Big(i, \prod_{\substack{p \leqq x \\ p \backslash q}} p\Big) = 1 \Leftrightarrow (i, q) = 1$$

(6) follows from it. By [5, p. 353] we have

$$\prod_{\substack{p \backslash q \\ p \geqq \log q}} \left(1 - \frac{1}{p}\right)^{-1} \leqq c_0$$

for every q. (7) follows from it. Using Mertens theorem we have (8).

We note that in this paper c_0 denotes a positive absolute constant and c_1, c_2, c_m, c'_m positive constants where the same letter way have different meaning.

Now, from (5) and (6) we have

$$N(q_i, q_j, t) \leqq \varphi(d_{ij}) \frac{\varphi(d_{ij}^o)}{d_{ij}^o} c_0 t d_{ij} q_{ij} \frac{\varphi(q_{ij})}{q_{ij}} \mathbf{A} \tag{9}$$

where

$$\mathbf{A} = \sum_{\substack{d \backslash d_{ij}^o \\ x = t d_{ij} q_{ij} / d}} \frac{|\mu(d)|}{\varphi(d) d} \frac{q_{ij}(x)}{\varphi(q_{ij}(x))}$$

Denote

$$x_{ij} = t d_{ij} q_{ij}$$

From (9) and by equality

$$\frac{\varphi(d_{ij})}{d_{ij}} \frac{\varphi(d_{ij}^o)}{d_{ij}^o} \frac{\varphi(q_{ij})}{q_{ij}} = \frac{\varphi(q_i)}{q_i} \frac{\varphi(q_j)}{q_j}$$

immediately follows

$$N(q_i, q_j, t) \leqq c_0 t \varphi(q_i) \varphi(q_j) \frac{q_{ij}(x_{ij})}{\varphi(q_{ij}(x_{ij}))} \sum_{d \backslash d_{ij}^o} \frac{|\mu(d)|}{\varphi(d) d} \mathbf{B} \tag{10}$$

where

$$B=\prod_{x<p\leqq x_{ij}}\left(1-\frac{1}{p}\right)^{-1}$$

If $x\geqq 2$ then by Mertens theorem

$$B\leqq c_1\frac{\log x_{ij}}{\log x}=c_1\left(\frac{\log x+\log d}{\log x}\right)\leqq c_1\left(1+\frac{\log d}{\log 2}\right)$$

If $x<2$ then $x_{ij}<2d$. Hence

$$B\leqq c_2\log 2d$$

In both cases the sum from (10) is not greater than an absolute positive constant. Summary

Theorem 3.

$$N(q_i,q_j,t)\leqq c_0 t\varphi(q_i)\varphi(q_j)\frac{q_{ij}(x_{ij})}{\varphi(q_{ij}(x_{ij}))}$$

Let us denote the following interval

$$I_{ij}=\left[\frac{1}{d_{ij}q_{ij}},\ \frac{\log q_{ij}}{d_{ij}q_{ij}}\right]$$

From Theorem 3 and Lemma 1 immediately follows

Theorem 4. If $t\notin I_{ij}$ then

$$N(q_i,q_j,t)\leqq c_0 t\varphi(q_i)\varphi(q_j)\tag{11}$$

If $t\in I_{ij}$ then

$$N(q_i,q_j,t)\leqq c_0 t\varphi(q_i)\varphi(q_j)\frac{\log\log 3q_{ij}}{\log 2td_{ij}q_{ij}}\tag{12}$$

Let us return to the proof of Theorem 2 now. Using conditions $t\notin I_{ij}$ or $t\in I_{ij}$ we can divide the sum $\Sigma N(q_i,q_j,t)$ $(m<i,j\leqq n)$ into two parts. By (11) the first part is not greater than the first term from (2). By (12) (or again by (11)) if $t=\log q_{ij}/d_{ij}q_{ij}$ then

$$N(q_i,q_j,t)\leqq c_0\frac{\log q_{ij}}{d_{ij}q_{ij}}\varphi(q_i)\varphi(q_j)$$

Using Cauchy inequality

$$\sum_{t\in I_{ij}}N(q_i,q_j,t)\leqq c_0\sum\sqrt{\varphi(q_i)\varphi(q_j)}\frac{\log q_{ij}}{d_{ij}q_{ij}}\sqrt{\varphi(q_i)\varphi(q_i)}\leqq$$

$$\leqq c_0\Big(\sum_{m<i\leqq n}\varphi(q_i)\Big)\sqrt{C}\tag{13}$$

where C is a partial sum of the convergent series (4) for $m<i,\ j\leqq n$. Thus (13) is the second term from (2). The proof is finished.

Some consequences

Theorem 6. If

$$\sum \frac{\varphi(q_i)}{q_i} < +\infty$$

then $\{q_i\}$ satisfies D.S.C. for every $\{f(q_i)\}$.

Theorem 7. Let $\{q_i\}$ be a sequence of relative primes positive integers. Then $\{q_i\}$ satisfies D.S.C. for every $\{f(q_i)\}$.

Proof. Let us divide the sequence $\{q_i\}$ into two subsequences such that the first contains every prime q_i and the second contains all the remaining ones. The first subsequence satisfies Duffin—Schaeffer's theorem [4, Theorem I]. For the second subsequences we have

$$\sum \frac{\log^2 q_i}{q_i} < +\infty$$

because every q_i contains two primes. From it follows (4) and therefore it again satisfies D.S.C.

Further we shall give two new criterions for sequances $\{q_i\}$ which satisfy D.S.C.

Theorem 8. Let $\{t_i^*\}$ be a finite sequence of all distances between neighbouring rational numbers from the interval [0, 1] which denominators (in a canonical form) are contained in the sequence $\{q_i\}_{i\leqq n}$. Let there for every sufficiently large n exist a finite sequence $\{t_i\}$ of positive real numbers such that

$$t_i \leqq t_i^*$$

for every i and

$$\sum_{t_i\leqq t} t_i \leqq c_0 t^2 \Big(\sum_{i\leqq n} \varphi(q_i)\Big)^2$$

for every $t>0$.

Then the sequence $\{q_i\}$ satisfies D.S.C. with every nonincreasing $\{f(q_i)\}$.

Let $N(\Sigma)$ denote the number of terms of indicated sum Σ.

Theorem 9. Let K be a fixed positive integer. Let there for infinitely many m exist a decomposition $\{q_i\}_{m<i}$ into K subsequences $\{q_i\}^j_{m<i}$, $j=1, 2, \ldots, K$ (for other m this decompositions are independent), for which there exist positive constants c^j_m and positive functions F^j_m such that

$$c^j_m \to 0 \quad \text{for} \quad m\to\infty \tag{14}$$

$$F^j_m \text{ is nondecreasing and } F^j_m(u)\to 0 \text{ for } u\to 0 \tag{15}$$

$$N(\Sigma)\leqq\Big(\sum_{m<i\leqq n}\varphi(q_i)\Big)F_m^j(\Sigma)+c_m^j\Big(\sum_{m<i\leqq n}\varphi(q_i)\Big) \tag{16}$$

for all sufficiently large n and for every subsum Σ of the sum of all distances between neighbouring rational numbers from the interval $[0, 1]$ which denominators are contained in $\{q_i\}^j_{m<i\leqq n}$ (or more precisely for every sequences of subsums $\{\Sigma_n\}$ for which $\Sigma_n\geqq c>0$).

Then the sequence $\{q_i\}$ satisfies D.S.C. with every nonincreasing $\{f(q_i)\}$.

By theory of the paper [1] the proofs of Theorems 8, 9 follow from following two criterions for quick sequences (see [1, Definitions 2,3, Corollary 1, Parts 4,5]).

Theorem 10. Let I be a fixed finite interval of real numbers. Let $\{y_i\}$ be a positively distributed (see [1, Definition 1]), one-to-one, infinite sequence of points from I. Order the finite sequence $\{y_i\}_{i\leqq n}$ on the increasing sequence

$$y_{i(1)}<y_{i(2)}<\ldots<y_{i(n)}$$

Put

$$\{t_i^*\}=\{y_{i(j+1)}-y_{i(j)}\}_{j<n}$$

Let there for every sufficiently large n exist a finite sequence $\{t_i\}$ such that

$$0<t_i\leqq t_i^* \tag{17}$$

and

$$\sum_{t_i\leqq t} t_i\leqq c_0t^2n^2 \tag{18}$$

Then $\{y_i\}$ is quick in I.

Theorem 11. Let $\{y_i\}$ be a positively distributed sequence. Let there for infinitely many m exist a decomposition $\{y_i\}_{m<i}$ into K subsequences $\{y_i\}^j_{m<i}$, $j=1, 2, \ldots, K$, for which ther exist positive constants c_m^j and positive functions F_m^j which satisfy (14), (15) and

$$N(\Sigma)\leqq(n-m)F_m^j(\Sigma)+c_m^j(n-m)$$

for all sufficiently large n and for every subsum Σ of the sum Σt_i^*.

Then $\{y_i\}$ is quick.

Proof of Theorem 10. Let $\{t_i\}$ be a finite nondecreasing sequence of positive real numbers for which

$$\sum_{t_i\leqq t} t_i\leqq c_1t^2 \tag{19}$$

for every $t>0$. (19) is equivalent to

$$\sum_{i\leqq j} t_i\leqq c_1t_j^2$$

for every $t_i \in \{t_i\}$. For $\{t_i\}$ we can construct a finite sequence $\{t_i'\}$ such that $\{t_i'\}$ is a positive root of the equation

$$c_1 t_i'^2 - t_i' - (t_1' + t_2' + \ldots + t_{i-1}') = 0 \tag{20}$$

Clearly

$$t_i' \leqq t_i \tag{21}$$

for every i and by (20) we have

$$1 = \frac{\sqrt{c_1}\, t_i'}{\sqrt{t_1' + t_2' + \ldots + t_i'}} \tag{22}$$

For $i = 1, 2, \ldots, j$ the sum of the right hands of (22) is a lower integral sum of the decreasing function $g(u) = \sqrt{c_1}/\sqrt{u}$ on the interval $[0, t_1' + t_2' + \ldots + t_j']$ which is divided into following intervals $[0, t_1']$, $[t_1', t_1' + t_2']$, $[t_1' + t_2', t_1' + t_2' + t_3']$ ... Thus this sum is not greater than the integral of $g(u)$ on $[0, t_1' + t_2' + \ldots + t_j']$. By computation we obtain

$$j \leqq 2\sqrt{c_1}\,\sqrt{t_1' + t_2' + \ldots + t_j'} \tag{23}$$

Let Σ be a subsum of the sum Σt_i. Since $\{t_i\}$ is nondecreasing and satisfies (21), it follows from (23)

$$N(\Sigma) \leqq 2\sqrt{c_1}\,\sqrt{\Sigma} \tag{24}$$

Also (24) is true if Σ is a subsum of the sum Σt_i^*, where $t_i \leqq t_i^*$ for every i.

Now, let $\{t_i\}$, $\{t_i^*\}$ be sequences from Theorem 10 which satisfy (17), (18). Put in (24) $c_1 = c_0 n^2$. Then

$$N(\Sigma) \leqq n 2\sqrt{c_0}\sqrt{\Sigma}$$

for every subsum Σ of the sum Σt_i^*. Thus $\{y_i\}$ satisfies the conditions in [1, Theorem 1]. Hence $\{y_i\}$ is quick.

Proof of Theorem 11. Theorem 11 is a generalization of [1, Theorem 2] and by its proof and notations

$$N(n) \geqq \max_{1 \leqq j \leqq K} \left\{ P^j(n) - m - (n - m) F_m^j \Big(\sum_{i > k} |I_i| \Big) - c_m^j (n - m) \right\} \tag{25}$$

where $P^j(n)$ denotes the number of terms from $\{y_i\}_{i \leqq n}^j$ for which

$$y_i \in I - \bigcup_{i>k} I_i$$

By assumption $\{y_i\}$ is a positive distributed sequence, then for all sufficiently large n

$$P^j(n) \geqq \frac{c}{K} |I - \cup I_i| n = c_1 n$$

for any j, $1 \leqq j \leqq K$, (j is dependent on n). By (14) for sufficiently large m, $c_m^j \leqq c_1/4$ for every j. By (15) for this m we can find k such that

$$F_m^j\left(\sum_{i>k} |I_i|\right) \leqq \frac{c_1}{4}$$

for every j. Thus by (25) $N(n) \geqq nc_1/2$.

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РЕЗЮМЕ

НЕКОТОРЫЕ НОВЫЕ ПРИЗНАКИ ДЛЯ ПОЛЕДОВА ТЕЛЬНОСТЕЙ УДОВЛЕТВОРЯЮЩИХ ГИПОТЕЗЕ ДАФФИНА–ШАФФЕРА, I.

Ото Штраух, Братислава

Пусть $\{q_i\}$ простая последовательность натуральных чисел и $\{f(q_i)\}$-последовательность положительных действительных чисел. По гипотезе Даффина—Шаффера из расходимости ряда $\Sigma\varphi(q_i)f(q_i)$ (φ — функция Эйлера) следует что диофантого неравенство

$$\left| u - \frac{p}{q_i} \right| < f(q_i)$$

имеет решение p для почти всех u, для бесконечного количества q_i, $(p, q_i) = 1$. В этой работе показано, что гипотеза Даффина—Шаффера имеет место если $(q_i, q_j) = 1$, $i \neq j$, или если ряд $\Sigma\varphi(q_i)/q_i$ сходится.

SÚHRN

NIEKTORÉ NOVÉ KRITÉRIÁ PRE POSTUPNOSTI, KTORÉ VYHOVUJÚ DUFFIN—SCHAEFFEROVEJ HYPOTÉZE, I.

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V práci je ukázané, že hypotéza Duffin—Schaeffera platí, ak postupnosť $\{q_i\}$ sa skladá z po dvoch nesúdeliteľných čísel, alebo ak konverguje rad $\Sigma\varphi(q_i)/q_i$. Ďalej sú tu odvodené dve nové kritériá pre platnosť hypotézy.