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A COHERENCE BETWEEN THE DIOPHANTINE APPROXIMATIONS AND THE DINI DERIVATES OF SOME REAL FUNCTIONS

OTO STRAUCH, Bratislava

Let $\{y_n\}$ be any one-to-one sequence of real numbers and let $\{z_n\}$ be any sequence of positive real numbers. Here the index n passes through the set of all positive integers. For these $\{y_n\}$, $\{z_n\}$ we define the real-valued function f on reals by the following

(1) $$f(y_n) = z_n \text{ for all } n \text{ and } f(x) = 0 \text{ for all } x \notin \{y_n\}.$$

This f is a generalization of the well known Riemann's elementary function. The following trivial coherence holds between the Dini derivates of f at x and the inequality

(2) $$|x - y_n| < z_n/z$$

Theorem 1. If the function f is defined by (1) and $z_n \to 0$, then it is true for its Dini derivates that:

$$D^+f(x) = \sup\{z > 0;\ (2)\ \&\ y_n > x \text{ holds for infinitely many } n\},$$

$$D_-f(x) = -\sup\{z > 0;\ (2)\ \&\ y_n < x \text{ holds for infinitely many } n\}$$

for all $x \notin \{y_n\}$, where $\sup \emptyset = 0$ and $-(+\infty) = -\infty$ (also $D^-f(x) = D_+f(x) = 0$, $D_-f(y_n) = D^-f(y_n) = +\infty$, $D_+f(y_n) = D^+f(y_n) = -\infty$).

Proof. It follows immediately from

$$\frac{f(x) - f(y_{k_n})}{x - y_{k_n}} = \frac{z_{k_n}}{y_{k_n} - x} > z > 0 \Leftrightarrow 0 < y_{k_n} - x < \frac{z_{k_n}}{z}$$

and from it that if the right hand holds for all k_n, then $y_{k_n} \to x$, since $z_{k_n} \to 0$. Evidently, Theorem 1 is true also for $\{z_n\}$ for which $z_{k_n} \to 0$ only for every bounded $\{y_{k_n}\}$ and also for $\{z_n\}$ for which there exists a constant $c > 0$ such that $\{y_n;\ z_n \geqq c\}$ is dense — in this case $D^+f(x)$, $D_-f(x)$ are infinite and it must be $D_-f(y_n) = +\infty$, $D^+f(y_n) = -\infty$.

I.e., if the function f is defined by (1) and $z_n \to 0$, then it is true:

f has at $x \notin \{y_n\}$ a derivative necessarily equal to zero if and only if for every $z > 0$ the inequality (2) holds only for finitely many n.

f has at $x \notin \{y_n\}$ at least one Dini derivate infinite if and only if for every $z > 0$ the inequality (2) holds for infinitely many n (it is true also for bounded $\{z_n\}$).

Now, using Theorem 1, we can prove some theorems of the inequalities (2) such that we apply on the function f, which is defined by (1), some general theorems of the Dini derivates. We shall do it in the Part I. of this paper.

On the other hand we can prove some theorems on the Dini derivates of functions f, which are defined by (1) and where $\{y_n\}$ is a rational sequence, such that we apply on the inequalities (2) some theorems of the diphantine approximations. We shall do it in the Part II. of this paper.

This method is used in the papers [1—5, 6—p. 374] and also in an unpublished paper of T. Šalát. We note that these papers are not containing the elementar Theorem 1 in an explicit form.

I.

For example, from Theorem 1 and from the following Theorems 2—5 follows immediately Theorem 6.

Theorem 2 (W. H. Young [7]). For any real function f the set of points at which f has at least one Dini derivate infinite is a G_δ set.

Theorem 3 (W. Sierpinski and A. N. Singh [8]). If the set of points of discontinuity of a real function f is dense, then the sets A^+, A^- are also dense and they have the power c. Here A^+ (A^-) denotes the set of points at which f has at least one right (left) Dini derivate infinite.

Theorem 4 (H. Lebesgue [9]). If a real function f has a finite variation on every finite interval, then f has a finite derivative almost everywhere.

Theorem 5 (O. Hájek [10]). For any real function f, the extreme bilateral derivatives $\underline{f}'$, $\bar{f}'$ must be of Baire class 2.

Theorem 6. For all $\{y_n\}$, $\{z_n\}$, $z_n \to 0$ it is true:

(3) The union

$\{y_n\} \cup \{x;$ for every $z > 0$, (2) holds for infinitely many $n\}$

is a G_δ set and hence if $\{y_n\}$ is dense, then the set

$\{x;$ there exists $z > 0$, (2) holds only for finitely many $n\}$

is of the first category.

(4) If $\{y_n\}$ is dense, then the following sets

$\{x;$ for every $z>0$, (2) & $y_n>x$ holds for infinitely many $n\}$
$\{x;$ for every $z>0$, (2) & $y_n<x$ holds for infinitely many $n\}$

are also dense and they have a power c.

(5) If the series $\Sigma z_n(y_n\in I)$ converges for every finite interval I, then the set

$\{x;$ there exists $z>0$, (2) holds for infinitely many $n\}$

is a nullset.

(6) For every $a>0$ the following sets are $G_{\delta\sigma}$.

$\{x;$ there exists z, $0<z<a$, (2) & $y_n>x$ holds only for finitely many $n\}-\{y_n\}$,
$\{x;$ there exists z, $z>a$, (2) & $y_n>x$ holds for infinitely many $n\}\cup\{y_n\}$.

Similarly for $y_n<x$.

We note that we can prove Theorem 6 directly, without to use the Theorems 1—5 and moreover by a short way and for any $\{z_n\}$ and $\{y_n\}$ need not be one-to-one (and in the definitions of the sets of (3), (6) the sequence $\{y_n\}$ need not be taken into consideration). It follows from this that applying the general Theorems 2—5 on the special function f, which is defined by (1), we choose from their proofs only a trivial part. From the following Denjoy—Young—Saks Theorem, which have a sufficiently complicated proof, we shall give a nontrivial result.

Theorem 7 (Denjoy—Young—Saks, see [11, p. 30]). For any real function f, with the possible exception of a nullset, the set of all reals can be decomposed into four sets:

$$X_1=\{x;\, D_-f(x)=D^-f(x)=D_+f(x)=D^+f(x)\neq\pm\infty\},$$
$$X_2=\{x;\, D^-f(x)=D_+f(x)\neq\pm\infty,\ D_-f(x)=-\infty,\ D^+f(x)=+\infty\},$$
$$X_3=\{x;\, D_-f(x)=D^+f(x)\neq\pm\infty,\ D^-f(x)=+\infty,\ D_+f(x)=-\infty\},$$
$$X_4=\{x;\, D_-f(x)=D_+f(x)=-\infty,\ D^-f(x)=D^+f(x)=+\infty\}.$$

Theorem 8. For all $\{y_n\}$, $\{z_n\}$, $z_n\to 0$, with the possible exception of a nullset, the set of all reals can be decomposed into two sets:

$X_1'=\{x;$ for every $z>0$, (2) holds only for finitely many $n\}$,
$X_2'=\{x;$ for every $z>0$, (2) & $y_n>x$ holds for infinitely many n and also (2) & $y_n<x$ holds for infinitely many $n\}$.

Proof. For every function f, which is defined by (1) and $z_n\to 0$, it is $X_3=X_4=\emptyset$ (if $z_n\not\to 0$, then $|X_3|=|X_4|=0$ only). By Theorem 1, $X_1=X_1'-\{y_n\}$ and $X_2=X_2'-\{y_n\}$. Thus, the proof is finished.

I.e.the following sets are nullsets;

$\{x;$ there exists $z'>0$, (2) holds only for finitely many n and there exists $z''>0$, (2) holds for infinitely many n},

{x; there exists $z>0$, (2) & $y_n>x$ holds only for finitely many n and (2) & $y_n<x$ holds for infinitely many n, or on the contrary}

for all $\{y_n\}$, $\{z_n\}$, $z_n\to 0$, and also the following sets are equal, except a nullset:

{x; f, which is defined by (1) from $\{y_n\}$, $\{z_n\}$, has at least one Dini derivate infinite at x},

{x; for $z=1$, (2) holds for infinitely many n},
{x; for $z=z_0$, (2) holds for infinitely many n},
{x; for every $z>0$, (2) holds for infinitely many n},
{x; there exists $z>0$, (2) holds for infinitely many n},
{x; for $z=1$, (2) & $y_n>x$ holds for infinitely many n},
{x; for $z=1$, (2) & $y_n<x$ holds for infinitely many n}.

Similary for finite derivative and for finitely many solutions.

At the end of the Part I., we note that Theorem 8 is true also for all sequences $\{y_n\}$, $\{z_n\}$, which are defined by the following:

Let $\{q_n\}$ be any one-to-one sequence of positive integers and let $\{g(q_n)\}$ be any sequence of positive real numbers which need not satisfy. $g(q_n)\to 0$. Then

(7) $\{y_n\}$ is equal to the one-to-one sequence of all rational numbers of the form p/q_n, where p is an integer and p, q_n are coprime. $\{z_n\}$ is defined such that if $y_n=p/q_i$, then $z_n=g(q_i)$.

By preceding and by Gallagher's Theorem in [12], for these $\{y_n\}$, $\{z_n\}$, one of the sets X_1', X_2' is always a nullset. In the general case it is not true.

We note that Theorem 8 also follows directly from Lemma 6 in [16, p. 26)].

II.

For any $\{q_n\}$, $\{g(q_n)\}$ we define the real-valued function f on reals by the following:

(8) $f(p/q_n)=g(q_n)$ for all rational p/q_n, p, q_n are comprime, and $f=0$ otherwise (i.e., f is defined by (1) from $\{y_n\}$, $\{z_n\}$, which are defined by (7)).

It follows immediately from Theorem 1 that the following two conjectures are equivalent:

Conjecture 1 (R. J. Duffin and A. C. Schaffer [13]). For all sequences $\{y_n\}$, $\{z_n\}$, which are defined by (7), it is true: For almost all x the inequality $|x-y_n|<z_n$ holds for infinitely many n, or for almost all x the inequality $|x-y_n|<z_n$ holds only for finitely many n, according as $\Sigma z_n(y_n\in I)$ diverges or converges for every finite interval I of a positive length, respectively.

Conjecture 2. For every function f, which is defined by (8), it is true: For almost all x the function f has at least one Dini derivate infinite, or for almost all x the function f has a zero derivative, according as $\Sigma f(x)(x \in I)$ diverges or converges for every finite interval I of a positive length, respectively.

Using Theorems 1,8 and the fact that the part connected with the convergence of this series is trivially true (by well known Borel—Cantelli Lemma), we can write the preceding conjectures by following:

For all sequences $\{y_n\}$, $\{z_n\}$ which are defined by (7) and for every function f which is defined by (1) from these $\{y_n\}$, $\{z_n\}$, it is true: If the series

$$\Sigma z_n(y_n \in [0, 1]) = \Sigma f(x)(x \in [0, 1]) = \Sigma \varphi(q_n) g(q_n) \quad (n = 1, 2, \ldots.)$$

(where φ denotes the Euler's function) diverges, then the sets $\{x$; for every $z > 0$, (2) & $y_n > x$ holds for infinitely many n and also (2) & $y_n < x$ holds for infinitely many $n\}$,

$$\{x;\, D^+ f(x) = +\infty,\, D_- f(x) = -\infty\}$$

have a full measure.

For example, the assertion of the Duffin—Schaeffer's conjecture is true for following sequences $\{q_n\}$, $\{g(q_n)\}$:

(9) $\{q_n\} = \{1, 2, 3, \ldots\}$, $\{q_n^2 g(q_n)\}$ is nonincreasing. It is equivalent to the well known Chiničin's Theorem in [14].

(10) There exists a constant $c > 0$ such that

$$\sum_{i \leqq n} q_i g(q_i) \leqq c \sum_{i \leqq n} \varphi(q_i) g(q_i)$$

for infinitely many n. It is the Duffin—Schaeffer's Theorem in [13] (From it follows (9). The condition (10) is true for $\{q_n\}$, for which $\varphi(q_n)/q_n \geqq c > 0$, here $\{g(q_n)\}$ is arbitrary.).

(11) $\{g(q_n)\} = \{c/q_n^2\}$, $\{q_n\}$ is arbitrary. It is the Erdös Theorem in [15] (By Theorem 8, it is sufficiently proved only for $\{1/q_n^2\}$.).

(12) $\{q_n\}$ satisfies

$$\frac{\varphi(q_n)}{\varphi(q_{n+1}} \leqq c < 1$$

for all sufficiently large n, $\{g(q_n)\}$ is arbitrary. It is result of the author of this paper, see [24]).

Some other results are obtained in the monograph V. G. Sprindžuk [16].

Now, for any $u > 0$ we define the real-valued function f_u on reals by the following:

(13) $f_u(p/q) = 1/q^u$ for every rational p/q and $f_u(x) = 0$ for all irrational x.

Here (and also in further) $q>0$, p are coprime integers. From Theorem 1 and from the following Theorems 9—13 of diophantine approximations follows immediately Theorem 14 on Dini derivates of f_u.

Theorem 9 (A. Hurwitz [17]). For every z, $0<z\leqq\sqrt{5}$ and for every irrational x the inequality

$$\left|x-\frac{p}{q}\right|<\frac{1}{zq^2} \tag{14}$$

holds for infinitely many p, q. For $x=(\sqrt{5}-1)/2$, $z>\sqrt{5}$ this inequality holds only for finitely many p, q.

Theorem 10 (A. A. Markov [18]). If for an irrational x there exists z, $0<z<3$ such that the inequality (14) holds only for finitely many p, q, then x is a quadratic irrational, i.e. has a periodic continued fraction (Conversely it is not true.).

Theorem 11. Any irrational x has bounded even (odd) partial quotients of a continued fraction of x if and only if there exists $z>0$ such that the inequality (14) holds only for finitely many p, q, $p/q>x(p/q<x)$.

We note that from Theorems 10, 8, 6 — part (3) and from the Chinčin's Theorem it follows immediately that for almost all x even and also odd partial quotients are unbounded and these x form a G_δ — residual set.

Theorem 12. For $z=1$ and for every irrational x the inequality (14) holds for infinitely many p, q, $p/q>x$ and it holds also for infinitely many p, q, $p/q<x$ (For $z>1$ it is not true.).

Theorem 13 (K. F. Roth [19]). Let x be an algebraic irrational number and $u>2$. Then the inequality

$$\left|x-\frac{p}{q}\right|<\frac{1}{q^u}$$

holds only for finitely many p, q.

Theorem 14.

(15) If $u<2$, then (by Theorem 12) the function f_u has at least one Dini derivate infinite everywhere (just then $D^+f_u(x)=+\infty$, $D_-f_u(x)=-\infty$ at every irrational x).

(16) The function f_2 is not differentiable (by Theorem 12) at every x. At every irrational x it has (by Theorems 9,12)

$$\max\{D^+f_2(x), |D_-f_2(x)|\}\geqq\sqrt{5}$$
$$\min\{D^+f_2(x), |D_-f_2(x)|\}\geqq 1$$

and at $x=(\sqrt{5}-1)/2$ this maximum is equal to $\sqrt{5}$. If this maximum is smaller than 3, then (by Markov's Theorem) x is quadratic irrational. The function f_2 has (by Chinčin's Theorem) at least one Dini derivate infinite almost everywhere and $D^+f_2(x)$ $(D_-f_2(x))$ is finite if and only if (by Theorem 11) x is irrational and it has a bounded even (odd) partial quotients.

(17) If $u>2$, then (by Chinčin's Theorem) the function f_u is almost everywhere differentiable (it has a zero derivative) and (by Roth's Theorem) it is differentiable at every algebraic irrational x, and it has at least one Dini derivate infinite at every Liouville's number.

(18) Every function f has (by Theorem 1) $D_-f_u=D^-f_u=+\infty$, $D_+f_u=D^+f_u=-\infty$ at every rational number and $D_+f_u=D^-f_u=0$ at every irrational number. Every function f_u has (by Young's Theorem) at least one Dini derivate infinite everywhere, except a set of the first category.

We note that some part of this theorem is contained in [3—5].

Again, using Theorem 1 and the following well known Theorem 15 of continued fractions, we can compute (Theorem 16) Dini derivates of f_2 at every irrational x.

Let us denote $[a_0; a_1, a_2, \ldots, a_n, \ldots]$ a continued fraction of x, $\{a_n\}$ are its partial quotients, $\{p_n/q_n\}$ are its convergents and $\{r_n\}$ are its complete quotients.

Theorem 15. For every irrational x it is true:

sup $\{z>0;$ (14) & $p/q>x$ holds for infinitely many $p, q\}=$ sup $\{z>0;$ (14) holds for infinitely many p_n, q_n, where n is odd$\}$.

Similarly for even n and $p/q<x$.

Just then see [20, p. 12, 32]

$$\left|x-\frac{p_n}{q_n}\right|=\frac{1}{q_n^2\left(r_{n+1}+\frac{q_{n-1}}{q_n}\right)}$$

$$r_{n+1}=[a_{n+1}; a_{n+2}, a_{n+3}, \ldots]$$

$$\frac{q_{n-1}}{q_n}=[0; a_n, a_{n-1}, \ldots, a_1]$$

Theorem 16.

$$D^+f_2(x)=\limsup_{n\to\infty}\,(a_{2n}+[0; a_{2n+1}, a_{2n+2}, \ldots]+[0; a_{2n-1}, a_{2n-2}, \ldots, a_1])$$

$$D_-f_2(x)=-\limsup_{n\to\infty}\,(a_{2n+1}+[0; a_{2n+2}, a_{2n+3}, \ldots]+[0; a_{2n}, a_{2n-1}, a_1])$$

for every irrational x.

The Dini derivates of f_2 have also following properties:

For any irrational x let us denote

$$L(x)=\sup\{z>0: (14) \text{ holds for infinitely many } p, q\} =$$
$$= \limsup_{n\to\infty} (a_n + [0; a_{n+1}, a_{n+2}, \dots] + [0; a_{n-1}, a_{n-2}, \dots, a_1]) =$$
$$= \max\{D^+f_2(x), |D_-f_2(x)|\}.$$

Theorem 17 (A. A. Markov [18], also see [21, p. 41]).

(19) $L(x)=\sqrt{9-4m^{-2}}$ for every x which is equivalent (i.e. which has the same continued fraction, except finitely many partial quotients) to a root of $F_m(x, 1)$, where $F_m(x, y)$ is a m-th Markov's binary quadratic form, its definition see [21, p. 31] (We note that two roots of $F_m(x, 1)$ are equivalent, see [21, p. 34].).

(20) $L(x)\geqq\sqrt{9-4m^{-2}}$ for every irrational x which is not equivalent to the roots of $F_1(x, F_2(x, 1), \dots, F_{m-1}(x, 1)$.

(21) If $L(x)<3$ then irrational x is equivalent to a root of some $F_m(x, 1)$.

Let us denote

$$L=\{L(x); x \text{ is irrational}\}.$$

The set L is caled the Lagrange's spectrum and it has following properties:

(22) The set L is closed and $L\subset[\sqrt{5}, +\infty)$.

(23) There exists a real constant μ_1 such that $[\mu_1, +\infty)\subset L$ and $[\mu_1 - u, +\infty)\not\subset L$ for all $u>0$. An estimate for μ_1 is $4{,}52<\mu_1\leqq 4{,}58$, see [22, p. 7, 77, 108].

(24) There exists a real constant $\mu_0>\sqrt{5}$ such that the Lebesgue's measure $|[\sqrt{5}, \mu_0]\cap L|=0$ and $|[\sqrt{5}, \mu_0+u]\cap L|>0$ for all $u>0$. An estimate for μ_0 is $3{,}28<\mu_0$, see [22, p. 79].

(25) The set L is disjoint with the following open intervals $(4\sqrt{30}/7, 10)$, $(4\sqrt{6}/3, \sqrt{689}/8)$, $(\sqrt{12}, \sqrt{13})$, see [22, p. 22].

Using Theorem 1 and some properties of continued fractions we can compute the Dini derivates of every function f_u, $u>2$, by following:

Theorem 18.

$$D^+f_u(x)=\limsup_{n\to\infty}\frac{a_{2n}}{q_{2n-1}^{u-2}}$$

$$D_-f_u(x) = -\limsup_{n\to\infty} \frac{a_{2n+1}}{q_{2n}^{u-2}}$$

for every irrational x.

Proof. For $u>2$ and irrational x it is again true: $D^+f_u(x) = \sup\{z>0;\ |x-p/q|<1/zq^u\ \&\ p/q>x$ holds for infinitely many $p, q\} = \sup\{z>0;\ |x-p_n/q_n|<1/zq_n^u$ holds for infinitely many odd $n\}$.

Similarly for $D_-f_u(x)$, even n and $p/q<x$. Just then

$$\left|x-\frac{p_n}{q_n}\right| = \frac{1}{q_n^u\left(\frac{r_{n+1}}{q_n^{u-2}}+\frac{q_{n-1}}{q_nq_n^{u-2}}\right)}$$

and

$$\frac{q_{n-1}}{q_nq_n^{u-2}} \to 0$$

since $q_{n-1}/q_n<1$ and $q_n>2^{(n-1)/2}$, see [20, p. 16]. Further

$$\frac{r_{n+1}}{q_n^{u-2}} = \frac{a_{n+1}}{q_n^{u-2}} + \frac{1}{r_{n+2}q_n^{u-2}}$$

and $r_{n+2}q_n^{u-2} \to +\infty$. Thus, the proof is finished.

We note that from the Theorems 1,11 immediately it follows that the following conjecture are equivalent:

Conjecture 3. Every algebraic x has bounded partialy quotients.

Conjecture 4. The function f_2 has bounded Dini derivates at every algebraic irrational x.

It follows from hence that it is sufficient for a proof of these conjectures to prove only that every algebraic irrational x is contained in the set of all x at which $f_u(u>2)$ has a zero derivate and at which f_2, after a limit transition $f_u \to f_2$, has a bounded Dini derivates. By the following, it is not simplication:

Let $\{p_1, p_2, \ldots, p_n\}$ be any finite set of primes. For this set and for any $u>0$, let us define the real-valued function $g_{n,u}$ on reals by

(26) $g_{n,u}(p/q) = 1/q^u$ for every rational p/q for which p, q have prime factors only from $\{p_1, p_2, \ldots, p_n\}$ (here $q>0$, p are again coprime integers) and $g_{n,u}=0$ otherwise.

From Theorem 1 and from the following Mahler's Theorem of diophantine approximations it follows immediately Theorem 20 on a derivative of $g_{n,u}$.

Theorem 19. (K. Mahler [23]). For all $u>0$ and for every algebraic irrational x the inequality

$$\left|x-\frac{p}{q}\right|<\frac{1}{q^u}$$

holds only for finitely many p, q, for which p, q have prime factors from $\{p_1, p_2, \ldots, p_n\}$ only.

Theorem 20. Every function $g_{n,u}$ has a zero derivative at every algebraic irrational x.

Now, let $\{p_n\}$ be a sequence of all primes. Then

$$g_{n,u} \rightarrow f_u (n \rightarrow \infty) \tag{27}$$

for every $u>0$. For this limit transition it is true:

(28) If $u<2$, then the algebraic irrationals are contained in the set of all points which are changing (at the limit transition (27)) from the points of a zero derivative on the points at which at least one Dini derivate is infinite.

(29) If $u>2$, then the algebraic irrationals are contained in the set of all points at which a zero derivative is conserve (at the limit transition (27)).

(30) If $u=2$ and Conjecture 4 is true, then the algebraic irrationals are contained in the set of all points which are changing (at the limit transition (27)) from the points of a zero derivative on the points of bounded Dini derivates.

By Young's Theorem, the function $g_{n,u}$ has at least one Dini derivate infinite everywhere, except a set of the first category and by Borel—Cautelli Lemma has a zero derivative almost everywhere.

Supplement. Here we give a simple proof of Theorem 15.

Proof. The equality from Theorem 15 follows from the following equality immediately

$$\begin{aligned} &\min \{q^2|x-p/q|;\ p/q \text{ lies between } p_{n-1}/q_{n-1} \text{ and } p_{n+1}/q_{n+1}\} = \\ &= \min \{q_{n-1}^2|x-p_{n-1}/q_{n-1}|,\ q_{n+1}^2|x-p_{n+1}/q_{n+1}|\}. \end{aligned} \tag{31}$$

Proof of (31). Clearly, if p/q lies between p_{n-1}/q_{n-1}, p_{n+1}/q_{n+1} and it is not equal to

$$\frac{p_n k + p_{n-1}}{q_n k + q_{n-1}} \tag{32}$$

for every integer k, $0 \leqq k \leqq a_{n+1}$, then p/q lies between two fractions in the form (32) and (see [20], p. 24) one of them has a smaller denominator than q and it lies more nearly to x than p/q. Thus $A(p/q) = q^2|x - p/q|$ has a minimum at p/q in the form (32). Just then

$$\left|x - \frac{p_n k + p_{n-1}}{q_n k + q_{n-1}}\right| = \left|\frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} - \frac{p_n k + p_{n-1}}{q_n k + q_{n-1}}\right| =$$

$$= \frac{r_{n+1} - k}{(q_n r_{n+1} + q_{n-1})(q_n k + q_{n-1})}.$$

I.e.

$$A(p/q) = \frac{(r_{n+1} - k)(q_n k + q_{n-1})}{(q_n r_{n+1} + q_{n-1})} = P(k)$$

for p/q in the form (32), where $P(k)$ is a quadratic polynomial for which $P(k) > 0$ for $0 \leqq k \leqq a_{n+1}$ and $P(k)$ has a positive discriminant. From it follows

$$\min_{0 \leqq k \leqq a_{n+1}} P(k) = \min \{P(0), P(a_{n+1})\}.$$

Thus, the proof is finished.

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Author's address:
Oto Strauch,
Katedra algebry a teórie čísel, MFFUK
Matematický pavilón,
Mlynská dolina
842 15 Bratislava

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РЕЗЮМЕ

СВЯЗЬ МЕЖДУ ТЕОРИЕЙ ДИОФАНТОВЫХ ПРИБЛИЖЕНИЙ И ПРОИЗВОДНЫМИ ЧИСЛАМИ НЕКОТОРЫХ ДЕЙСТВИТЕЛЬНЫХ ФУНКЦИЙ

Ото Штраух, Братислава

В первой части показывается (теорема 8), с применением теоремы Данжуа о производных числах, что для почти всех вещественных чисел x существует одновременно бесконечное число решений, в целых несократимых p, $q>0$, следующих диофантовых неравенств:

$$0<x-\frac{p}{q}<f(q),\quad 0<\frac{p}{q}-x<f(q),$$

$$\left|x-\frac{p}{q}\right|<f(q),\quad \left|x-\frac{p}{q}\right|<cf(q).$$

В второй части найденны (теорема 16), с применением теории цепных дробей, производные числа функции, которая равная $1/q^2$ в рациональных точках p/q и нулю в иррациональных.

SÚHRN

SÚVIS DIOFANTICKÝCH APROXIMÁCIÍ S DERIVOVATEĽNOSŤOU NIEKTORÝCH REÁLNYCH FUNKCIÍ

Oto Strauch, Bratislava

Práca sa skladá z dvoch častí, I, II. V časti I sa nachádza obecná veta opisujúca tento súvis a pomocou nej, z viet o derivovateľnosti reálnych funkcií, sú odvodzované vety o aproximáciach. V časti II sú z viet o diofantických aproximáciach odvodzované vety o derivovateľnosti špeciálnych reálnych funkcií.