UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLII—XLIII—1983

AN APPLICATION OF CRAMER'S RULE ON A CLASS OF LINEAR MAPS

PAVEL KOSTYRKO, Bratislava

In the present paper we shall deal with linear transformations of an algebra R over a field F. We shall encounter the algebra in the following way: We are given an associative and commutative ring R and a subfield F of R. Then we can consider R as an algebra over F by taking $x+y$ usual sum in R, and αx and xy, $\alpha \in F$, $x, y \in R$, to be the ring product of α and x, and x and y in R, respectively.

A derivation f in the algebra R over F is a mapping of R into R such that conditions

$$f(x+y)=f(x)+f(y), \quad f(\alpha x)=\alpha f(x) \tag{0}$$

and

$$f(xy)=f(x)y+xf(y) \tag{1}$$

are fulfilled for each $x, y \in R$ and $\alpha \in F$. Some of properties of derivations are in general well-known and are introduced in basic monographs (see e.g. [1], p. 167). It is easy to verify, that if f is a derivation then

$$f(x^n)=nx^{n-1}f(x) \tag{n}$$

holds for each integer $n>1$ and each $x \in R$ (we put $mz=z+z+\ldots+z$ m times if m is a positive integer, $0.z=0$ and $mz=-(z+\ldots z)$ $-m$ times if m is a negative integer). The aim of the present paper is to prove the following

Theorem. Let R be an associative and commutative ring which has the characteristic 0 and the identity e, and let F be a subfield of R. Let $f\colon R\to R$ be a linear transformation of R over F, i.e. f fulfils (0). If there exists $n>1$ such that (n) holds, then f is a derivation, i.e. f fulfils (1).

Proof. Further we shall also use short $m=me$ for an integer m. The validity of our theorem for $n=2$ is an immediate consequence of the identity $4xy=(x+y)^2-(x-y)^2$. We can prove (1) by applying of f on the identity and to use (0) and (2).

Let us suppose $n>2$. We show that (0) and (n) imply (2), what is sufficient to the proof of our theorem. We shall use the next easy provable facts: $f(0)=f(e)=0$.

The identity $n(x+re)^{n-1}f(x) = f((x+re)^n)=f\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}r^k\right)$ holds for each $x\in R$ and $r=0, 1, \ldots, n-2$. We can adapt this system of identities on the system (S) of $n-1$ equations with $n-1$ unknowns $f(x^n)$, $f(x^{n-1})$, ..., $f(x^2)$ from R. The system (S) has coefficients in F.

$$\binom{n}{0}f(x^n)+r\binom{n}{1}f(x^{n-1})+\ldots+r^k\binom{n}{k}f(x^{n-k})+\ldots+$$

$$+r^{n-2}\binom{n}{n-2}f(x^2)=n\left(\sum_{k=0}^{n-2}r^k\binom{n-1}{k}x^{n-k-1}\right)f(x), \qquad (S)$$

for $r=0, 1, \ldots, n-2$. The determinant d of the matrix S of the system (S) is easy to count by transforming it on Vandermonde determinant.

$$d=|S|=\begin{vmatrix} \binom{n}{0} & 0 & \ldots & 0 & \ldots & 0 \\ \binom{n}{0} & \binom{n}{1} & \ldots & \binom{n}{k} & \ldots & \binom{n}{n-2} \\ \vdots & \vdots & & \vdots & & \vdots \\ \binom{n}{0} & r\binom{n}{1} & \ldots & r^k\binom{n}{k} & \ldots & r^{n-2}\binom{n}{n-2} \\ \vdots & \vdots & & \vdots & & \vdots \\ \binom{n}{0} & (n-2)\binom{n}{1} & \ldots & (n-2)^k\binom{n}{k} & \ldots & (n-2)^{n-2}\binom{n}{n-2} \end{vmatrix} =$$

$$=\prod_{k=0}^{n-2}\binom{n}{k}\prod_{0\leqslant j<i\leqslant n-2}(i-j)\neq 0.$$

Replace the last column of the matix S by the right side of the system (S) and count the determinant d_0 of this new matrix S_0. The determinant d_0 can be expressed as a sum of $n-1$ determinants, from which only one is not vanishing.

$$d_0=|S_0|=\begin{vmatrix} \binom{n}{0} & 0 & \ldots & 0 & 0 \\ \binom{n}{0} & \binom{n}{1} & \ldots & \binom{n}{n-3} & \binom{n-1}{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{n}{0} & r\binom{n}{1} & \ldots & r^{n-3}\binom{n}{n-3} & r^{n-2}\binom{n-1}{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{n}{0} & (n-2)\binom{n}{1} & \ldots & (n-2)^{n-3}\binom{n}{n-3} & (n-2)^{n-2}\binom{n-1}{n-2} \end{vmatrix} nxf(x)$$

$=2dxf(x)$. On the other hand if we replace the last column of S by the left side of the system (S) we have $d_0 = |S_0| = df(x^2)$. Hence $f(x^2) = 2xf(x)$, and the proof is complete.

Remark. It is easy to verify that the method of the proof of our Theorem can be also used to prove the next more general

Theorem'. Let R be an associative and commutative ring which has the characteristic 0 and the identity, let A be a subring of R, and let F be a subfield of A. Let $f\colon A \to R$ be a linear transformation of A into R over F. If there exists $n > 1$ such that (n) holds for each $x \in A$, then f is a derivation.

REFERENCE

[1] Jacobson, N.: Lectures in abstract algebra III, Springer-Verlag, New York Heidelberg Berlin.

Received: 12. 6. 1981

Author's address:
Pavel Kostyrko
Katedra algebry a teórie čísel MFFUK
Mlynská dolina
842 15 Bratislava

SÚHRN

APLIKÁCIA CRAMEROVHO PRAVIDLA NA JEDNU TRIEDU LINEÁRNYCH ZOBRAZENÍ

Pavel Kostyrko, Bratislava

V práci sa dokazuje nasledujúce tvrdenie: Nech R je asociatívny a komutatívny okruh charakteristiky 0 s jednotkou, nech A je podokruh a nech F je podteleso A. Nech $f\colon A \to R$ je lineárne zobrazenie A do R nad F. Ak existuje $n > 1$ tak, že $f(x^n) = nx^{n-1}f(x)$ platí pre každé $x \in A$, tak f je derivácia, t. j. $f(xy) = f(x)y + xf(y)$ platí pre každé $x, y \in A$.

РЕЗЮМЕ

ПРИМЕНЕНИЕ ПРАВИЛА КРАМЕРА ДЛЯ ОДНОГО КЛАССА ЛИНЕЙНЫХ ОТОБРАЖЕНИЙ

Павел Костырко, Братислава

В работе доказывается утверждение: Пусть R — ассоциативное и коммутативное кольцо характеристики нуль обладающее единицей, пусть A — подкольцо R и пусть F является подтелом A. Пусть $f\colon A \to R$ линейно отображает A в R над F. Если существует $n > 1$ так что $f(x^n) = nx^{n-1}f(x)$ имеет место для всякого $x \in A$, то f является дифференцированием, т. е. $f(xy) = f(x)y + xf(y)$ верно для каждого x и y из A.