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ON WEAKLY CLOSED FUNCTIONS

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R. F. Dickman [1] recently introduced the following notion: *A* real function $f: R \to R$ is said to be weakly closed at some point $x_0 \in R$ provided for each sequence $\{x_i\}_{i=1}^{\infty}$ of points in R which converges to x_0 the set of members of the sequence $\{f(x_i)\}_{i=0}^{\infty}$ is closed; a function f is said to be weakly closed when it is weakly closed at each $x_0 \in R$. It turns out that weak closedness along with some other properties implies the continuity of functions (cf. [1]).

The present paper is initiated by the following problem: For arbitrary $f: R \to R$ denote by A_f the set of all $x_0 \in R$ at which f is not weakly closed; what can be said about A_f? We begin with the following two simple examples.

Example 1. Define $f: R \to R$ by $f(x)=0$ for $x \in Q$ (Q is always the set of rational numbers) and $f(x)=e^x$ otherwise. It is easily verified that $A_f = Q$.

Example 2. Let D be a closed nowhere dense subset of R. Let $g: R \to [-1, 1]$ be defined by $g(x) = \sin[1/\text{dist}(x, D)]$ for $x \in R \setminus D$, and $g(x)=0$ otherwise. It can be easily verified that $A_g = D$.

Note that each point of A_f, where f is the function from Example 1, is a removable point of the weak non-closedness in the sense that by a single change of the value of f at arbitrary $z \in Q$ we obtain a function f^* which is weakly closed at z (we put $f^*(z)=e^z$, and $f^*(x)=f(x)$ for $x \neq z$). On the other hand, for g in Example 2, no $x \in A_g$ is a removable point of weak non-closedness.

The preceding considerations lead us to the following notion: For each $f: R \to R$ define ess A_f such that $x \in \text{ess } A_f$ provided no function $f': R \to R$ such that $f'(y) = f(y)$ for each $y \neq x$, is weakly closed at x. Clearly ess $A_f \subset A_f$.

Now the following theorems give a characterization of sets A_f and ess A_f. In fact, A_f can be an arbitrary set (Theorem 1), but the sets ess A_f for various f can be characterized as the F_σ sets (Theorems 2 and 3).

Theorem 1. For each $A \subset R$ there is some $f: R \to R$ with $A_f = A$.

Proof. If $A = R$ put $f(x) = x$ for $x \in Q$, $f(x) = x+1$ for $x \in R \setminus Q$. Clearly $A_f = R$.

Now assume that $A \neq R$. Let B be the interior of A, and let $C = A \setminus B$. Let

$B=\bigcup_{n=1}^{\infty} I_n$ where I_n are pairwise disjoint open intervals $I_n=(a_n, b_n)$ or $I_n=\emptyset$. Denote $c_n=\min\{|I_n|, 1\}$, where $|I|$ denotes the length of I. Choose some $a\notin A$ and define a function f as follows:

$$f(x)=x \quad \text{if} \quad x\notin A \tag{1}$$

and for $x\in A$ put

$$f(x)=\begin{cases} x-c_n/2^n & \text{if} \quad x=a_n \text{ or } x=b_n, \\ \varphi_n(x) & \text{if} \quad x\in I_n\cap Q, \\ \psi_n(x) & \text{if} \quad x\in(I_n\setminus Q), \\ a & \text{otherwise,} \end{cases}$$

where φ_n, ψ_n are continuous functions $I_n\rightarrow R$ with the following properties:

$$\text{Both } \varphi_n \text{ and } \psi_n \text{ are strictly increasing in } I_n\,; \tag{2}$$

$$\varphi_n(x)<\psi_n(x) \quad \text{for each} \quad x\in I_n\,; \tag{3}$$

$$\lim \varphi_n(x)=\lim \psi_n(x)=f(a_n) \quad \text{when} \quad x\rightarrow a_{n^+} \quad \text{and}$$

$$\lim \varphi_n(x)=\lim \psi_n(x)=f(b_n) \quad \text{when} \quad x\rightarrow b_{n^-}\,; \tag{4}$$

$$|f(x)-x|<1/2^{n-1} \quad \text{for each} \quad x\in I_n\,. \tag{5}$$

The last condition is satisfied when the graphs of φ_n and ψ_n are near the segment connecting the points $\langle a_n, f(a_n)\rangle$ and $\langle b_n, f(b_n)\rangle$ in the plane. Note that

$$f(x)\neq x \quad \text{for} \quad x\in C. \tag{6}$$

We show that $A=A_f$. Let $x_0\in A$. If $x_0\in B$ then for some n, $x_0\in I_n$; if $x_0\in I_n\cap Q$ let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points in $I_n\setminus Q$ converging to x_0. Then the set of members of the sequence $\{f(x_i)\}_{i=0}^{\infty}=\{\varphi(x_0), \psi(x_1), \psi(x_2), \psi(x_3), \ldots\}$ is not closed, by (2) and (3), and consequently f is not weakly closed at x_0. If $x_0\notin Q$ the argument is similar.

If $x_0\in C$ there is a sequence $x_i\notin A$ such that $\lim x_i=x_0$. Moreover, since by (6) $f(x_0)\neq x_0$, we may choose the sequence such that $x_i\neq f(x_0)$ for each i. By (1) x_0 is a limit point of the sequence $\{f(x_i)\}_{i=1}^{\infty}$, but x_0 is not a member of the sequence $\{f(x_i)\}_{i=0}^{\infty}$, i.e. f is not weakly closed at x_0.

Finally assume that $x_0\notin A$. We show that f is weakly closed at x_0. Let $\lim x_i=x_0$. Let $P_1\cup P_2\cup P_3\cup P_4=P$ be a decomposition of the set P of positive integers such that $n\in P_1$ iff $x_n\notin A$, $n\in P_2$ iff $x_n=a_k$ or $x_n=b_k$ for some k, $n\in P_3$ iff $x_n\in C$ and $n\notin P_2$, and $n\in P_4$ iff $x_n\in B$. It suffices to show that

$$B_i=\{f(x_n);\, n\in P_i\}\cup\{f(x_0)\}$$

is a closed set for $i=1, 2, 3, 4$. When some P_i is finite then the corresponding set B_i is clearly closed. So assume that P_1, P_2, P_3, P_4 are infinite sets. Then $B_1=$

$\{x_n; n \in P_1\} \cup \{x_0\}$ is closed since $\lim x_n = x_0$ (see also (1)). Similarly $B_3 = \{a, f(x_0)\}$ is a finite set hence it is closed. The set B_2 contains numbers of the form $x_n - c_{k(n)}/2^{k(n)}$, and x_0, and since $\lim x_n - c_{k(n)}/2^{k(n)} = x_0$ for $n \to \infty$, $n \in P_2$, B_2 is closed.

Thus it remains to consider the set B_4. Without loss of generality we may assume that $x_n > x_0$ for $n \in P_4$ (otherwise we divide P_4 into two subsets). If $x_0 = a_m$ for some m, then $x_n \in I_m$ whenever $n \in P_4$ is sufficiently large, hence by (4) $\lim f(x_n) = f(x_0) = a_m$ for $n \to \infty$, $n \in P_4$ and B_4 is closed.

If x_0 is not the left-hand endpoint of some I_m, let for $k \in P_4$ $x_k \in I_{n(k)}$. Since by (1) $f(x_0) = x_0$, for $k \in P_4$ we have

$$|f(x_k) - f(x_0)| \leq |f(x_k) - x_k| + |x_k - x_0|. \tag{7}$$

Now by (5), $|f(x_k) - x_k| \leq 1/2^{n(k)-1}$ and since $\lim n(k) = \infty$ for $k \to \infty$, $k \in P_4$, the right-hand side of (7) tends to 0 whenever $k \to \infty$, $k \in P_4$, i.e. $\lim f(x_k) = f(x_0)$ for $k \to \infty$, $k \in P_4$, and B_4 is closed. This finishes the proof of Theorem 1.

The following lemma will be useful in the proof of Theorem 2. Note that in the following by limit point we always mean a finite limit point.

Lemma. Let $f: R \to R$ be a function. Then $a \in \operatorname{ess} A_f$ if and only if there is a sequence $\alpha = \{a_n\}_{n=1}^{\infty}$, $a_n \in R$, which converges to a, and such that the set $f(\alpha)$ of members of the sequence $\{f(a_n)\}_{n=1}^{\infty}$ has at least two different limit points which do not belong to $f(\alpha)$.

Proof. When for some $\alpha = \{a_n\}_{n=1}^{\infty}$ the set $f(\alpha)$ has limit points $u, v \notin f(\alpha)$, $u \neq v$, then $f(\alpha) \cup \{f(a)\}$ cannot be closed and hence $a \in \operatorname{ess} A_f$.

Conversely assume that each $f(\alpha)$ has at most one limit point which does not belong to $f(\alpha)$. Let $\alpha = \{a_n\}_{n=1}^{\infty}$ and $\beta = \{b_n\}_{n=1}^{\infty}$ be sequences converging to a. Let u be a limit point of $f(\alpha)$ and v a limit point of $f(\beta)$ and let $u \notin f(\alpha)$, $v \notin f(\beta)$. Assume that $u \neq v$. Consider the sequence $\{a_1, b_1, a_2, b_2, \ldots\}$; when we omit all members x such that $f(x) = u$ or $f(x) = v$, we obtain again an infinite sequence $\gamma = \{c_n\}_{n=1}^{\infty}$ such that u, v are limit points of $f(\gamma)$ and $u, v \notin f(\gamma)$, and this is a contradiction. Hence there is a number u_0 such that for each sequence $\alpha = \{a_n\}_{n=1}^{\infty}$ converging to a, $a_n \neq a$, the set $f(\alpha) \cup \{u_0\}$ is closed. Now it suffices to define $f^*(x) = f(x)$ for $x \neq a$, $f^*(a) = u_0$, and we obtain a function f^* which is weakly closed at a. Consequently, $a \notin \operatorname{ess} A_f$, q.e.d.

Theorem 2. For each $f: R \to R$ the set $\operatorname{ess} A_f$ is of the type F_σ.

Proof. Fix some f. For each positive integers m, k, denote by $M(m, k)$ the set of those $x \in R$, for which the following holds: There is a sequence $\alpha = \{x_n\}_{n=1}^{\infty}$ converging to x such that the set $f(\alpha) = \{f(x_n)\}_{n=1}^{\infty}$ has two different limit points $p(x)$, $q(x)$ not belonging to $f(\alpha)$ and such that

$$|p(x) - q(x)| \geq 1/m, \quad |p(x)| \leq k, \quad |q(x)| \leq k.$$

By Lemma clearly ess $A_f = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} M(m, k)$. So it suffices to show that each set $M(m, k)$ is closed. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points from $M(m, k)$ converging to some a. We show that $a \in M(m, k)$. For each n choose corresponding $p(a_n)$, $q(a_n)$. Since both sequences $\{p(a_n)\}_{n=1}^{\infty}$ and $\{q(a_n)\}_{n=1}^{\infty}$ are bounded there exists a subsequence $\{a_{n(i)}\}_{i=1}^{\infty}$ such that $\{p(a_{n(i)})\}_{i=1}^{\infty}$ and $\{q(a_{n(i)})\}_{i=1}^{\infty}$ are convergent sequences. We may assume without loss of generality that $\{a_n\}_{n=1}^{\infty}$ is this subsequence. Denote $\lim p(a_n) = p$ and $\lim q(a_n) = q$. Clearly

$$|p-q| \geq 1/m, \ |p| \leq k, \ |q| \leq k. \tag{8}$$

Moreover, we may assume that for each n, $|a - a_n| < 1/n$, $|p(a_n) - p| < 1/n$, and $|q(a_n) - q| < 1/n$.

Denote by $\alpha^n = \{x_i^n\}_{i=1}^{\infty}$ a sequence converging to a_n such that $p(a_n)$, $q(a_n)$ are limit points of $f(\alpha^n)$, and $p(a_n), q(a_n) \notin f(\alpha^n)$. For each n choose integers i(1) and i(2) such that

$$|a_n - x_{i(1)}^n| < 1/n, \ |f(x_{i(1)}^n) - p(a_n)| < 1/n, \ f(x_{i(1)}^n) \neq p, q,$$

and similarly for $x_{i(2)}^n$ and $q(a_n)$. Note that such i(1), i(2) must exist. Now

$$|x_{i(1)}^n - a| \leq |x_{i(1)}^n - a_n| + |a_n - a| < 2/n$$

and similarly for i(2), i.e.

$$\lim x_{i(1)}^n = \lim x_{i(2)}^n = a \ \text{ for } \ n \to \infty.$$

Moreover

$$|p - f(x_{i(1)}^n)| \leq |p - p(a_n)| + |p(a_n) - f(x_{i(1)}^n| < 2/n$$

and similarly

$$|q - f(x_{i(2)}^n)| < 2/n.$$

Thus p, q are limit points of $f(\alpha)$, where $\alpha = \{x_{i(1)}^1, x_{i(2)}^1, x_{i(1)}^2, x_{i(2)}^2, \ldots\}$ is a sequence converging to a, and $p, q \notin f(\alpha)$. By (8), $a \in M(m, k)$, q.e.d.

Theorem 3. Let $A \subset R$ be a F_σ set. Then there is a bounded function f such that ess $A_f = A$, and f is continuous outside the set A.

Proof. Let G be the interior of A. Then $B = A \setminus G$ is a nowhere dense F_σ set of dimension 0 hence there are pairwise disjoint (nowhere dense) closed sets A_i such that $B = A_1 \cup A_2 \ldots$ (cf. [2], p. 254—255). Let I_1, I_2, ... be the connected components of G (the sequence can be finite or infinite). For each n let φ_n, ψ_n be continuous functions from I_n to the open interval $(0, 1/n)$ such that $\varphi_n(x) < \psi_n(x)$ for each x, and let both φ_n, ψ_n vanish at the endpoints of I_n. Define $h_0 \colon R \to [0, 1)$ by

$$h_0(x)=\begin{cases}\varphi_n(x) & \text{for} \quad x\in I_n\cap Q,\\ \psi_n(x) & \text{for} \quad x\in I_n\setminus Q,\\ 0 & \text{for} \quad x\notin G.\end{cases}$$

It is easy to see that ess $A_{h_0}=G$, and that h_0 is continuous outside the set G.

For each n let g_n be the function g from Example 2 for $D=A_n$. Clearly ess $A_{g_n}=A_n$. Put $h_n(x)=g_n(x)/3^n$, for each $x\in R$. Then

$$\sup_x |h_n(x)|=1/3^n. \tag{9}$$

Now let

$$f(x)=\sum_{n=0}^{\infty} h_n(x) \text{ for each } x\in R.$$

Clearly f is bounded. Let $x_0\notin A$, and let U_k be a neighbourhood of x_0 disjoint with $A_1, \ldots, A_k$. Then for $x\in U_k$

$$f(x)=h_0(x)+\sum_{i>k} h_i(x)=h_0(x)+d(x),$$

since $h_i(x)=0$ for $x\notin A_i$. But by (9) $|d(x)|\leq 1/2.3^k$ and h_0 is continuous at x_0. Consequently f is continuous at x_0 (and hence f is weakly closed at x_0).

Finally let $x_0\in A$. If $x_0\in G$, then $x_0\in$ ess $A_{h_0}\subset$ ess A_f since $f(x)=h_0(x)$ for $x\in G$. So assume that $x_0\in A_n$ for some n. Then similarly as in the preceding case, since A_i are disjoint, $\lim h_i(x)=0$ for $i=0, 1, \ldots, n-1$, when $x\to x_0$ and $\limsup h_n(x)=1/3^n$, $\liminf h_n(x)=-1/3^n$ for $x\to x_0$. So

$$\limsup_{x\to x_0} f(x)\geq 3^{-n}-\sum_{k\geq 1} 3^{-n-k}=1/2.3^n$$

and similarly

$$\liminf_{x\to x_0} f(x)\leq -1/2.3^n.$$

Now it is easy to see that the range of f at x_0 contains the interval $(-1/2.3^n, 1/2.3^n)$, and hence $x_0\in$ ess A_f, q.e.d.

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РЕЗЮМЕ

О СЛАБО ЗАМКНУТЫХ ФУНКЦИЯХ

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В статье характеризуются множества точек слабой замкнутости вещественных функций.

SÚHRN

O SLABO UZAVRETÝCH FUNKCIÁCH

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V práci sú charakterizované množiny bodov slabej uzavretosti reálnych funkcií.