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ON DISCONTINUITY POINTS OF FUNCTIONS OF SOME CLASSES

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In the paper [2] Z. Grande has proved that a set $A \subset R_2$ is the set of all discontinuity points of a certain linearly (separately) continuous function f: $R_2 \to R$ if and only if A is an F_σ-set of the first Baire category contained in the Cartesian product of two linear sets each of which is a set of the first Baire category in R.

In connection with the mentioned result we shall investigate the similar question for the class of all quasicontinuous functions and the class of all cliquish functions f: $R_2 \to R$.

Remember that if X, Y are topological spaces, then the function f: $X \to Y$ is said to be quasicontinuous at the point $p \in X$ if for each neighbourhood $V(f(p)) \subset Y$ of the point $f(p)$ and each neighbourhood $U(p) \subset X$ of the point p there exists such a non-void open set $G \subset U(p)$ that $f(G) \subset V(f(p))$. The function f: $X \to Y$ is said to be quasicontinuous (on X) if it is quasicontinuous at each point of the space X (cf. [3], [5]). Clearly, each continuous function is quasicontinuous, too.

Let X be a topological and (Y, τ) a metric space. The function f: $X \to Y$ is said to be cliquish at the point $p \in X$ if for each $\varepsilon > 0$ and each neighbourhood $U(p)$ of the point p there exists such a non-void open set $G \subset U(p)$ that for each two points $x, y \in G$ we have

$$\tau(f(x), f(y)) < \varepsilon$$

The function f: $X \to Y$ is said to be cliquish on X if it is cliquish at each point of X (cf. [6], [7]).

If X is a topological space and (Y, τ) a metric space, then each function f: $X \to Y$ which is quasicontinuous (on X) is cliquish (on X), too. The converse is not true (cf. [6]).

From the mentioned result of Grande we obtain at once the following

Theorem 1. Let f: $R_2 \to R$ be quasicontinuous on R_2. Then the set $D(f)$ of all discontinuity points of f is an F_σ-set of the first Baire category in R_2. Conversely, if $M \subset R_2$ is an F_σ-set, $M \subset A \times B$, A, B are linear sets of the first Baire category in

R, then there exists such a function f: $R_2 \to R$ that f is quasicontinuous on R_2 and $D(f) = M$.

Proof. The first part of Theorem is a well-known fact (cf. [5]).

Let $M \subset R_2$ fulfil the assumptions of Theorem. On account of the mentioned result of Grande there exists such a linearly continuous function f: $R_2 \to R$ that $D(f) = M$. Since f is linearly continuous, it is linearly quasicontinuous, too (i.e. for each fixed $x_0(y_0)$ the function $f(x_0, y)$ $(f(x, y_0))$ is quasicontinuous on R). But then $f(x, y)$ is quasicontinuous on R_2 on the basis of a well-known result of S. Kempisty (cf. [3]). Theorem follows.

In connection with Theorem 1 we shall prove the following result. Let us remark that the topological space X is said to be a Baire space if each non-void open set $M \subset X$ is a set of the second Baire category in X (cf. [1]).

Theorem 2. (i) Let X be a topological space and f: $X \to R$ be cliquish (on X). Then the set $D(f)$ is an F_σ-set of the first Baire category in X.

(ii) If X is a Baire topological space and $M \subset X$ is an F_σ-set of the first Baire category in X, then there exists such a function f: $X \to R$ cliquish on X that $D(f) = M$.

Proof. The part (i) of the foregoing theorem is a known fact (cf. [5]). We prove the part (ii). Let $M = \bigcup_{n=1}^{\infty} M_n$, where M_n $(n = 1, 2, \ldots)$ are closed sets. We may assume that $M_1 \subset M_2 \subset M_3 \ldots$. Since X is a Baire space and M is a set of the first category, the set $X - M$ is dense in X. But then each set $X - M_n$ $(n = 1, 2, \ldots)$ is dense in X. Since M_n $(n = 1, 2, \ldots)$ is closed in X, the set M_n $(n = 1, 2, \ldots)$ is nowhere dense in X.

Define the function f: $X \to R$ in the following way: $f(x) = 0$ for $x \in X - M$, $f(x) = 1$ for $x \in M_1$, $f(x) = \frac{1}{k}$ for $x \in M_k - M_{k-1}$ $(k = 2, 3, \ldots)$.

We shall prove that $D(f) = M$.

Let $p \in M$. Then $f(p) > 0$. Since M is a set of the first Baire category and each neighbourhood of p is a set of the second Baire category, each neighbourhood of p contains a point $x \in X - M$ with $f(x) = 0$. Hence $p \in D(f)$.

Let $p \in X - M$. Let $\varepsilon > 0$. Choose a positive integer m such that

$$\frac{1}{m} < \varepsilon \tag{1}$$

Since $p \notin M_k$ $(k = 1, 2, \ldots, m)$ we can take such a neighbourhood $U(p)$ of the point p that

$$U(p) \cap M_k = \emptyset \quad (k = 1, 2, \ldots, m). \tag{2}$$

According to the definition of the function f in view of (1), (2) we have

$|f(x)-f(p)|<\frac{1}{m}<\varepsilon$ for each $x\in U(p)$. Hence f is continuous at the point p. Hence we have $D(f)=M$.

We prove that f is cliquish on X. Let $p\in X$. If $p\notin M$, then f is continuous at the point p and so it is cliquish at p, too. Let $p\in M$. Let $\varepsilon>0$. Choose a positive integer k such that

$$\frac{1}{k}<\varepsilon \tag{3}$$

Let $U(p)$ be an arbitrary neighbourhood of p. Since the sets M_j $(j=1, 2, \ldots, k)$ are nowhere dense in X, there exists such a non-void open set $G\subset U(p)$ that

$$G\cap M_j=\emptyset \qquad (j=1, 2, \ldots, k)$$

(cf. [4], p. 37). But then for each $x\in G$ we have $0\leqq f(x)<\frac{1}{k}$ and so for each two points y, z from G we get on account of (3)

$$|f(y)-f(z)|<\frac{1}{k}<\varepsilon$$

Hence f is cliquish at p. This ends the proof.

Finally we shall formulate a problem about characterizing sets of discontinuity points of quasicontinuous functions.

The mentioned result of Grande (cf. [2]) characterizes the sets of discontinuity points of linearly continuous functions $f\colon R_2\to R$. Our Theorem 2 gives a characterization of the sets of discontinuity points of cliquish functions $f\colon R_2\to R$. The question arises to characterize the sets of discontinuity points of quasicontinuous functions $f\colon R_2\to R$ (or even $f\colon X\to R$, X is a topological space).

Denote by L, K, and Q the class of all linearly continuous, quasicontinuous and cliquish functions $f\colon R_2\to R$. Then we have $L\subset K\subset Q$. On the basis of the mentioned characterizations of sets of discontinuity points of functions from classes L and Q it can be expected that the sets of discontinuity points of functions from the class K will be chaacterized by the property to be F_σ-sets, to be sets of the first Baire category and by a certain property P, which is weaker than the property to be included in the Cartesian product of two linear sets each of which is a set of the first Baire category in R.

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SÚHRN

O BODOCH NESPOJITOSTI FUNKCIÍ ISTÝCH TRIED

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V práci sa skúma otázka charakterizácie množín bodov nespojitosti funkcií kvázispojitých a kľukatých (sliquish). Táto otázka je úplne rozriešená pre reálne kľukaté funkcie definované na Baireových topologických priestoroch.

РЕЗЮМЕ

ОБ ТОЧКАХ РАЗРЫВА ФУНКЦИЙ НЕКОТОРЫХ КЛАССОВ

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В работе исследуется вопрос характеризации множеств точек разрыва квазинепрерывных функций и извилистых (cliquish) функций. Вопрос вполне разрешен для действительных извилистых функций определенных на топологических пространствах Бера.