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ON POINTS OF ABSOLUTE CONTINUITY OF CONTINUOUS FUNCTIONS

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The notion of points of absolute continuity of functions is introduced in the papers [4], [5]. The point $p \in [a, b]$ is said to be a point of absolute continuity of the function $f: [a, b] \to R$ if there is such a $\delta > 0$ that the function f is absolutely continuous on the interval $I = [a - \delta, a + \delta] \cap [a, b]$.

Denote by $G(f)$ the set of all points of absolute continuity of the function $f: [a, b] \to R$. Put $N(f) = [a, b] - G(f)$. It is easy to see that for an arbitrary function f the set $G(f)$ is open in $[a, b]$ (in the relative topology of $[a, b] \subset R$), hence $N(f)$ is a closed subset of $[a, b]$.

The properties of sets $G(f)$, $N(f)$ for various classes of real functions are investigated in the paper [4]. For example it is shown there that if f is a continuous function of finite variation in $[a, b]$, then $N(f)$ is a perfect set, further if f is differentiable on $[a, b]$, then $N(f)$ is a nowhere dense set in $[a, b]$. The converse question is studied in the paper [6], it is proved there that to an arbitrary perfect set $N \subset [a, b]$ there exists such a continuous function of finite variation in $[a, b]$ that $N = N(f)$ and to an arbitrary nowhere dense set $N \subset [a, b]$ there exists such a differentiable (on $[a, b]$) function f that $N = N(f)$.

In this paper we shall complete the mentioned results, we shall give a new proof for the nowhere density of $N(f)$ in the case of differentiable function f, we shall generalize this result and investigate the structure of the metric space $C(a, b)$ of all continuous functions on $[a, b]$ (with the sup-metric) from the viewpoint of points of absolute continuity of functions from $C(a, b)$. Finally we shall call attention to corollaries of the proved results in the theory of Lipschitzian points of functions (cf. [1]).

At first we shall give a new proof of the following result from [4]. The proof of this result in [4] is based on a certain result from [8] about points of the uniform differentiability of functions.

Theorem A. If $f \in C(a, b)$ is differentiable on (a, b), then $N(f)$ is a nowhere dense set in $[a, b]$.

Proof. Denote by C and D the set of all continuity points and discontinuity points of the function f', respectively. Since f' is a function in the first Baire class, the set D is a set of the first Baire category in $[a, b]$ (cf. [9], p. 182).

If $x \in C$, then f' is bounded in a certain neighbourhood of x. From this it follows easily that the function f fulfils in an interval containing x the Lipschitzian condition, hence $x \in G(f)$. Therefore $C \subset G(f)$ and so

$$N(f) \subset D \cup \{a\} \cup \{b\} \tag{1}$$

Since D is a set of the first category, on account of (1) $N(f)$ is such a set, too. Since $N(f)$ is a closed set in $[a, b]$, it must be nowhere dense. This ends the proof.

Our method of proving Theorem A enables us to generalize the foregoing result in such a way that the assumption of differentiability of f will be replaced by a weaker assumption.

Theorem 1. Let $f \in C(a, b)$, let f be an approximatively differentiable function on (a, b). Then $N(f)$ is a nowhere dense set in $[a, b]$.

Proof. Denote by C, D the set of all continuity points and the set of all discontinuity points of the function f'_{ap} in $[a, b]$, respectively. Since f'_{ap} is a function in the first Baire class ([3], p. 152), the set D is a set of the first Baire category in $[a, b]$.

Let $x \in C$. Then there exists such an interval $I = [x - \delta, x + \delta] \subset [a, b]$ that f'_{ap} is bounded on I, i.e. there exists such a $K > 0$ that for each $y \in I$ we have

$$|f'_{ap}(y)| \leqq K \tag{2}$$

For arbitrary two points x_1, x_2 from I we have (cf. [3], p. 158):

$$|f(x_1) - f(x_2)| = |f'_{ap}(y)|\ |x_1 - x_2| \tag{3}$$

where y is a number between x_1 and x_2. It follows from (2), (3) that

$$|f(x_1) - f(x_2)| \leqq K |x_1 - x_2|,$$

hence f is Lipschitzian on I. Therefore $C \subset G(f)$ and so $N(f) \subset D \cup \{a\} \cup \{b\}$. Further we proceed similarly as in the proof of Theorem A.

The function $f\colon [a, b] \to R$ is said to have the property (V) if the following holds: Let

$$E = \left\{x \in [a, b];\ f(x) - \frac{f(b) - f(a)}{b - a}\, x > \frac{bf(a) - af(b)}{b - a}\right\},$$

$$F = \left\{x \in [a, b];\ f(x) - \frac{f(b) - f(a)}{b - a}\, x < \frac{bf(a) - af(b)}{b - a}\right\}.$$

Then we have

$$\bar{D}_r f(x) \geqq D_l f(x) \quad \text{for } x \in E$$

and

$$\underline{D}_r f(x) \leqq \bar{D}_l f(x) \quad \text{for } x \in F,$$

where

$$\underline{D}_r f(x) = \lim_{h\to 0+} \inf \frac{f(x+h)-f(x-h)}{2h},$$

$$\bar{D}_r f(x) = \lim_{h\to 0+} \sup \frac{f(x+h)-f(x-h)}{2h};$$

similarly $\underline{D}_l f(x)$, $\bar{D}_l f(x)$ are defined through replacing $h\to 0+$ by $h\to 0-$.

Theorem 2. Let the function $f\in C(a, b)$ be symmetrically differentiable on the interval (a, b) and have the property (V). Then the set $N(f)$ is nowhere dense in $[a, b]$.

Proof. For each $x\in(a, b)$ there exists (finite)

$$f^{(s)}(x) = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$$

From this it is easy to see that the symmetric derivative $f^{(s)}$ of the function $f\in C(a, b)$ is a function in the first Baire class, since $f^{s}(x) = \lim_{n\to\infty} f_n(x)$, where

$$f_n(x) = \frac{f\left(x+\frac{1}{n}\right)-f\left(x-\frac{1}{n}\right)}{\frac{2}{n}} \qquad (n=1, 2, \ldots).$$

are continuous functions on (a, b).

Further we can proceed similarly as in the proof of Theorem 1 taking into consideration the fact that for the symmetric derivative of a function fulfilling the assumptions of Theorem the mean value theorem holds, too (cf. [9], Theorem 4).

In what follows we shall study the structure of the class $A(a, b)$ of all such functions $f\in C(a, b)$ for which $G(f)\neq\emptyset$ holds. Put $B(a, b)=C(a, b)-A(a, b)$. Hence $B(a, b)$ is the class of all such functions f from $C(a, b)$ for which $N(f)=[a, b]$.

Let us remark that $A(a, b)$ is a dense set in $C(a, b)$ since all Lipschitzian functions on $[a, b]$ belong to the class $A(a, b)$.

Theorem 3. The class $A(a, b)\subset C(a, b)$ is an $F_{\sigma\delta\sigma}$-set of the first Baire category in $C(a, b)$.

Corollary. The set $B(a, b)\subset C(a, b)$ is a $G_{\delta\sigma\delta}$-set, residual in $C(a, b)$.

Proof. Denote by Q the set of all rational numbers of the interval (a, b). Let $q\in Q$, $\delta>0$, $\varepsilon>0$, $\eta>0$. Denote by $A(q, \delta, \varepsilon, \eta)$ the class of all such functions $f\in C(a, b)$ for which the following holds:

If

$$\{(a_i, b_i); \qquad i=1, 2, \ldots, m\}$$

is an arbitrary finite system of non-overlapping intervals with

$$(a_i, b_i) \subset [q-\delta, q+\delta] \qquad (i=1, 2, \ldots, m),$$

$$\sum_{i=1}^{m} (b_i - a_i) \leqq \eta, \quad \text{then} \quad \sum_{i=1}^{m} |f(b_i) - f(a_i)| \leqq \varepsilon.$$

We shall show that $A(q, \delta, \varepsilon, \eta)$ is a closed subset of the space $C(a, b)$.

Let $f_k \in A(q, \delta, \varepsilon, \eta)$ $(k=1, 2, \ldots)$, let the sequence $\{f_k\}_{k=1}^{\infty}$ converge to a function f from $C(a, b)$. We shall show that $f \in A(q, \delta, \varepsilon, \eta)$.

Let $\{(a_i, b_i);\ i=1, 2, \ldots, m\}$ be an arbitrary system of non-overlapping intervals, let

$$(a_i, b_i) \subset [q-\delta, q+\delta] \qquad (i=1, 2, \ldots, m),$$

$$\sum_{i=1}^{m} (b_i - a_i) \leqq \eta.$$

Since the convergence $f_k \to f$ in $C(a, b)$ is the uniform convergence, we can choose such a p that for each $x \in [a, b]$ and an arbitrarily chosen $v > 0$ we have

$$|f_p(x) - f(x)| \leqq \frac{\varepsilon}{2vm} \tag{4}$$

Then a simple estimation yields

$$\sum_{i=1}^{m} |f(b_i) - f(a_i)| \leqq \sum_{i=1}^{m} |f(b_i) - f_p(b_i)| +$$

$$+ \sum_{i=1}^{m} |f_p(b_i) - f_p(a_i)| + \sum_{i=1}^{m} |f_p(a_i) - f(a_i)| \tag{5}$$

Since $f_p \in A(q, \delta, \varepsilon, \eta)$, the second summand on the right-hand side of (5) is not greater than ε. Further the first and also the third summand on account of (4) is not greater than $\frac{\varepsilon}{2v}$. Therefore we get from (5) the estimation

$$\sum_{i=1}^{m} |f(b_i) - f(a_i)| \leqq \varepsilon\left(1 + \frac{1}{v}\right) \tag{6}$$

Since the inequality (6) is true for an arbitrary $v > 0$, we get from this by $v \to \infty$ the inequality

$$\sum_{i=1}^{m} |f(b_i) - f(a_i)| \leqq \varepsilon$$

Hence $f \in A(q, \delta, \varepsilon, \eta)$ and so the set $A(q, \delta, \varepsilon, \eta)$ is closed.

It follows from the provious result that the set

$$A\left(q, \frac{1}{n}\right) = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A\left(q, \frac{1}{n}, \frac{1}{j}, \frac{1}{k}\right)$$

is an $F_{\sigma\delta}$-set in $C(a, b)$ and so

$$A(q) = \bigcup_{n=1}^{\infty} A\left(q, \frac{1}{n}\right)$$

is an $F_{\sigma\delta\sigma}$-set in $C(a, b)$. But obviously we have

$$A(a, b) = \bigcup_{q \in Q} A(q)$$

and in view of the countability of Q the set $A(a, b)$ is an $F_{\sigma\delta\sigma}$-set in $C(a, b)$.

Further, if $f \in A(a, b)$, then the function f is absolutely continuous on a certain interval $I \subset (a, b)$ and therefore it has almost everywhere in I the finite derivative (cf. [10], p. 403). It follows from the foregoing that $A(a, b)$ is a subset of the set H of all such functions from $C(a, b)$ which have at least at one point of (a, b) a finite derivative. But H is a set of the first Baire category (cf. [7], p. 260), hence $A(a, b)$ is such a set, too. This ends the proof.

The function $f \in C(a, b)$ is said to be strongly locally Hölderian at the point $p \in (a, b)$ if there exist such a $\delta > 0$ and the real numbers $\alpha > 0$, $K > 0$ that for arbitrary two points $x, y \in (p - \delta, p + \delta)$ we have $|f(x) - f(y)| \leqq K|x - y|^{\alpha}$. The function $f \in C(a, b)$ is said to be a locally Hölderian function at the point $p \in (a, b)$ if there exist such a $\delta > 0$ and the real numbers $\alpha > 0$, $K > 0$ that for each $x \in (p - \delta, p + \delta)$ we have

$$|f(x) - f(p)| \leqq K|x - p|^{\alpha}$$

Denote by $P^*(f)$, $P(f)$ $(f \in C(a, b))$ the set of all such points $p \in (a, b)$ at which the function f is strongly locally Hölderian and locally Hörderian, respectively. Obviously we have $P^*(f) \subset P(f)$.

If we put in the previous definitions $\alpha = 1$, we obtain the notion of strongly locally Lipschitzian and locally Lipschitzian (at a point) functions, respectively (cf. [1], [2]).

Denote by $L^*(f)$ and $L(f)$ the set of all such points $p \in (a, b)$ at which the function f is strongly locally Lipschitzian and locally Lipschitzian, respectively. Evidently we have

$$L(f) \subset P(f), \quad L^*(f) \subset P^*(f)$$

From the proofs of Theorem 1 and Theorem 2 we can easily deduce the following results.

Theorem 1′. If $f \in C(a, b)$ is an approximately differentiable function on (a, b), then the set $[a, b] - L^*(f)$ is a nowhere dense set in $[a, b]$.

Corollary. If $f \in C(a, b)$ is approximately differentiable on (a, b), then each of the sets

$$[a, b] - P(f),\ [a, b] - P^*(f),\ [a, b] - L(f)$$

is a nowhere dense set in $[a, b]$.

Theorem 2′. If $f \in C(a, b)$ is symmetrically differentiable on (a, b) and has the property (V), then the set $[a, b] - L^*(f)$ is nowhere dense in $[a, b]$.

Corollary. If $f \in C(a, b)$ is symmetrically differentiable on (a, b) and has the property (V), then each of the sets

$$[a, b] - P(f),\ [a, b] - P^*(f),\ [a, b] - L(f)$$

is nowhere dense in $[a, b]$.

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SÚHRN

O BODOCH ABSOLÚTNEJ SPOJITOSTI SPOJITÝCH FUNKCIÍ

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Práca úzko nadväzuje na prácu [4] a dopĺňa ju. V práci je opísaná štruktúra množín bodov absolútnej spojitosti pre niektoré triedy funkcií.

РЕЗЮМЕ

ОБ ТОЧКАХ АБСОЛЮТНОЙ НЕПРЕРЫВНОСТИ НЕПРЕРЫВНЫХ ФУНКЦИЙ

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Работа узко примыкается к работе [4] и дополняет её. В работе описано строение множеств точек абсолютной непрерывности для некоторых классов функций.