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# ORDER-PRESERVING MAPPINGS OF COUNTABLE DENSE SETS OF REALS

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## 1. Introduction and the result

A well-known theorem says that every countable dense ordered sets without the first and the last element have the same order type, i.e. there is an order-preserving bijective mapping of one of them onto the other (see [2]).

In the case that these sets are subsets of the set $R$ of reals, of course also some further properties of that mapping can be required. The main concern is so that it can be extended into a continuous function on the whole $R$. We can see that here the dense order is not enough, but the density of both sets in $R$ (or another additional condition) must be required. Similar problems are studied in [1].

In the paper it is demonstrated that in such a case there is a bijection between both sets which is not only continuous but which has all derivatives as well and even we can develop it into the MacLaurin series. A function which can be developped to the MacLaurin series (which converges to the function for every real $x$) will be called analytic function.

The main result of the present paper is:

**1.1. Theorem.** Let $A$ and $B$ be dense countable subsets of $R$. Let $\varphi(x)$ be a continuous positive function and let $k$ be a positive integer. Then there exists an increasing analytic function $f(x)$ which is a one to one correspodence between the sets $A$ and $B$ and satisfies the inequality

$$\max\{|f(x)-x|, |f'(x)-1|, |f''(x)|, \ldots, |f^{(k)}(x)|\} < \varphi(x) \tag{1.1}$$

for every $x \in R$.

This result can be extended as follows; $N$ denote the set of positive integers.

**1.2. Theorem.** Let $(A_i)_{i\in N}$, $(B_i)_{i\in N}$ be two collections of mutually disjoint, countable, dense subsets of the real line. Let $\varphi(x)$ be a continuous positive function and $k \in N$. The there exists an analytic increasing function $f(x)$ satisfying (1.1) and such that

$$f(A_i)=B_i \text{ for all } i\in N. \tag{1.2}$$

In Section 3 we shall show that we cannot ask for $f$ to be a polynomial instead to be an analytic function in the theorem above. Analogously, the dense subsests of the real line cannot be replaced by dense subsets of the complex plane.

## 2. Proof of the theorems

**2.1. Lemma.** Let $\Psi(x)$ be a continuous positive function. Then there exists an even analytic function $h(x)$ satisfying the inequality

$$h(x)<\psi(x) \text{ for every real } x. \tag{2.1}$$

**Proof.** We shall construct the function $h(x)$ in the form $h(x)=e^{-H(x)}$. To construct the function $H(x)$ define

$$F_n=\frac{1}{\min\{\Psi(x):|x|\leqq 2^{n+1}\}} \tag{2.2}$$

and for $i<n$ denote by $q(n, i)$ the least positive integer such that

$$(2^{n-i})^{q(n,i)}\geqq 2^{n-i}\frac{F_n}{F_i}.$$

Further define $k(1)=0$ and

$$k(n)=2\max\{k(n-1)+1,\ q(n,1),\ \dots,\ q(n,n-1)\}$$

for all integers $n>1$.

All the integers $k(n)$ are obviously even and $k(n)<k(n+1)$ for all $n\in N$.

Now define

$$c_{k(n)}=\frac{F_n}{2^{n.k(n)}} \text{ for all } n\in N$$

and

$$H(x)=\sum_{n=1}^{\infty} c_{k(n)}.x^{k(n)}. \tag{2.3}$$

We have to prove that the series (2.3) converges for every real $x$. Let $x\in R$. There is $m\in N$ such that $|x|\leqq 2^m$. For all integers $j>m$ we obtain

$$|c_{k(j)}x^{k(j)}|\leqq c_{k(j)}.2^{m.k(j)}\leqq\frac{F_j}{2^{j.k(j)}}\cdot 2^{m.k(j)}=$$

$$= \frac{F_j}{2^{(j-m)k(j)}} \leqq \frac{F_j}{2^{(j-m)q(j,m)}} \leqq \frac{F_j}{2^{(j-m)} \cdot \frac{F_j}{F_m}} = (F_m \,.\, 2^m) \,.\, 2^{-j}.$$

Hence (2.3) converges by the comparative criterion. Since all $k(n)$ are even and positive, the function $H(x)$ is even and hence

$$h(x) = \mathrm{e}^{-H(x)},$$

is even, too. The function $h(x)$ is obviously positive ad analytic.

It remains to show $h(x) < \Psi(x)$ for every $x \in R$.

Let $x \in R$, $|x| < 2$. Since $0 < H(0) \leqq H(x)$ and $\mathrm{e}^{-x} < x^{-1}$ for all positive $x$, we have

$$h(x) = \mathrm{e}^{-H(x)} \leqq \mathrm{e}^{-H(0)} < \frac{1}{H(0)} = \frac{1}{c_{k(1)}} = \frac{1}{F_1} \leqq \Psi(x).$$

Let $x \in R$, $|x| \geqq 2$. Then there is $m \in N$ such that $2^m \leqq |x| < 2^{m+1}$. From this we obtain

$$c_{k(m)} \,.\, 2^{m \,.\, k(m)} \leqq c_{k(m)} \,.\, |x|^{k(m)} \leqq \sum_{i=1}^{\infty} c_{k(i)} x^{k(i)} = H(x).$$

This implies

$$h(x) = \mathrm{e}^{-H(x)} < \frac{1}{H(x)} \leqq \frac{1}{c_{k(m)} \,.\, 2^{m \,.\, k(m)}} = \frac{1}{F_m} \leqq \Psi(x).$$

**2.2. Remark.** It is easy to see that the function $h(x)$ is increasing on the interval $(-\infty, 0)$.

**2.3. Lemma.** For every positive continuous function $\Psi(x)$ and every positive integer $k$ there is a positive analytic function $g(x)$ such that

$$\max\{g(x), |g'(x)|, \ldots, |g^{(k)}(x)|\} < \Psi(x) \,.\, \mathrm{e}^{-x^2} \tag{2.4}$$

for every real $x$.

**Proof.** By 2.1 and 2.2 we may assume that $\Psi(x)$ is an even continuous positive function, increasing on the interval $(-\infty, 0)$ and such that $\Psi(x) < \frac{1}{2}$ for every real $x$. By lemma 2.1. there is an even analytic positive function $g_0(x)$ satisfying the inequality

$$g_0(x) < \Psi(x) \,.\, \mathrm{e}^{-(k+1)x^2} \text{ for every real } x.$$

For $1 \leqq n \leqq k$ define

$$g_n(x)=\frac{1}{2^n}\cdot\int_{-\infty}^{-x} g_{n-1}(t)\,\mathrm{d}t\,.\int_{-\infty}^{w} g_{n-1}(t)\,\mathrm{d}t \tag{2.5}$$

and denote

$$G_{n-1}(x)=\int_{-\infty}^{x} g_{n-1}(t)\,\mathrm{d}t\,. \tag{2.6}$$

We show that $g_n(x)$ is an even analytic positive function satisfying the inequality

$$\max\{g_n(x),\,|g_n'(x)|,\,\ldots,\,|g_n^{(n)}(x)|\}<\Psi(x).\mathrm{e}^{-(k-n+1)x^2} \tag{2.7}$$

for every real $x$.

This we can prove by induction.

For $n=0$ it is obvious.

Let now $g_n(x)$ be an even analytic positive function satisfying (2.7). Since

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x=\sqrt{\pi}<2$$

and

$$0<g_n(x)<\Psi(x).\mathrm{e}^{-(k-n+1)x^2<1/2\mathrm{e}^{-x^2}},$$

$\int_{-\infty}^{\infty} g_n(t)\,\mathrm{d}t$ converges and

$$0<\int_{-\infty}^{w} g_n(t)\,\mathrm{d}t<\int_{-\infty}^{\infty} g_n(t)\,\mathrm{d}t<1 \quad \text{for every real } x\,. \tag{2.8}$$

(2.5), (2.6) and (2.8) implies that $g_{n+1}(x)$ is an even positive function.

The function $g_n(x)$ is analytic, hence the function $G_n(x)$ is also analytic and therefore the function $g_{n+1}(x)$ is analytic, too.

Now we shall verify the inequality (2.7).

Let $x$ b *a* nonpositive real. The functions $\Psi(x)$ and $\mathrm{e}^{-(k-n)x^2}$ are continuous and increasing on $(-\infty, 0)$, and $\mathrm{e}^{-x^2}>0$, hence according to the mean value theorem we have

$$0<\int_{-\infty}^{x} g_n(t)\,\mathrm{d}t\leqq\int_{-\infty}^{x} \Psi(t).\mathrm{e}^{-(k-n)t^2}.\mathrm{e}^{-t^2}\,\mathrm{d}t\leqq$$

$$\leqq\max\{\Psi(t).\mathrm{e}^{-(k-n)t^2}: t\in(-\infty, x)\}.\int_{-\infty}^{x} \mathrm{e}^{-t^2}\,\mathrm{d}t<\Psi(x).\mathrm{e}^{-(k-n)x^2}.$$

Since all the functions are even, the inequality holds for positive $x$, too. Therefore

$$0<G_n(x)<\Psi(x).\mathrm{e}^{-(k-n)x^2} \quad \text{for every real } x\,. \tag{2.9}$$

From (2.5), (2.6), (2.8) and (2.9) now we have

$$g_{n+1}(x)=\frac{1}{2^{n+1}}\cdot G_n(-x).G_n(x)<\Psi(x).e^{-(k-(n+1)+1)x^2}$$

for every real $x$.

Further we get (using (2.6) and (2.8))

$$|g'_{n+1}(x)|=\frac{1}{2^{n+1}}|-G'_n(-x).G_n(x)+G_n(-x).G'_n(x)|<$$

$$<2^{-n}.g_n(x)<\Psi(x).e^{-(k-n)x^2}.$$

Finally, let $0\leqq i\leqq n$. Then we obtain

$$|g_{n+1}^{(i+1)}(x)|=\frac{1}{2^{n+1}}|[-g_n(-x).G_n(x)+g_n(x).G_n(-x)]^{(i)}|\leqq$$

$$\leqq\frac{1}{2^{n+1}}\sum_{j=0}^{i}\binom{i}{j}g_n^{(i-j)}(x).[G_n^{(j)}(x)+G_n^{(j)}(-x)]|<\Psi(x).e^{-(k-n)x^2}.$$

We have used that $\sum_{j=0}^{i}\binom{i}{j}=2^i$ and that $|G_n^{(j)}(x)|<1$. The inequality (2.7) is proved.

Now we put

$$g(x)=g_k(x). \tag{2.9}$$

Then $g(x)$ is an analytic positive function satisfying (2.4).

**2.4. Lemma.** Let $\varphi(x)$ be a continuous positive function. Let $g(x)$ be *a* continuous function such that

$$|g(x)<\varphi(x).e^{-x^2} \text{ for every real } x.$$

Let $P_n(x)$ be a polynomial of degree $n$. Then the function $\frac{P_n(x).g(x)}{\varphi(x)}$ is bounded.

**Proof.** Since $|g(x)|<\varphi(x).e^{-x^2}$, we have

$$\left|\frac{P_n(x).g(x)}{\varphi(x)}\right|<|P_n(x)|.e^{-x^2},$$

and the function on the right is obviously bounded.

**2.5. Lemma.** Let $\varphi(x)$ be a continuous positive function and let $F(x)$ be a function. Let $n$ and $k$ be positive integers. Let $C=\{c_1, \ldots, c_n\}\subset R$ and $D$ be a dense subset of $R$, let $c\notin C$. Then there exists an analytic function $G(x)$ which vanishes at the points $c_1, \ldots, c_n$ and such that

$$F(c)+G(c)\in D, \tag{2.10}$$

$$\max\{|G(x)|, |G'(x)|, \ldots, |G^{(k)}(x)|\} < \frac{\varphi(x)}{2^n}. \tag{2.11}$$

**Proof.** By lemma 2.3 there is an analytic positive function $g$ satisfying (2.4). Define a function $h(x)$ by

$$h(x) = (x - c_1) \ldots . (x - c_n) . g(x). \tag{2.12}$$

Therefore $h(x) = P_n(x) . g(x)$, where $P_n(x)$ is a polynomial of degree $n$. Then

$$h^{(j)}(x) = \sum_{i=0}^{j} \binom{j}{i} P_n^{(i)}(x) g^{(j-i)}(x)$$

for all nonnegative integers $j$. By lemma 2.4. the functions

$$\frac{P_n^{(i)}(x) . g^{(j-i)}(x)}{\varphi(x)}$$

are bounded for all nonnegative $i$, $j$, $i \leqq j \leqq k$, therefore the functions

$$\frac{h(x)}{\varphi(x)}, \ldots, \frac{h^{(k)}(x)}{\varphi(x)}$$

are bounded, too.

Let $L$ be their common bound and

$$p = \frac{1}{2^n . L}.$$

The set $D$ is a dense subset of $R$, hence there is

$$d \in D \cap (F(c) - p . h(c), F(c) + p . h(c)).$$

Let $r = \dfrac{d - F(c)}{h(c)}$. Define $G(x) = r . h(x)$ for all real $x$.

The function $G(x)$ is obviously analytic, it vanishes at all the points $c_1, \ldots, c_n$ and $F(c) + G(c) = d \in D$. Finally, since $|r| < p$, for nonnegative integer $j \leqq k$ we get

$$\left|\frac{G^{(j)}(x)}{\varphi(x)}\right| = |r| \cdot \left|\frac{h^{(j)}(x)}{\varphi(x)}\right| < p . L = \frac{1}{2^n}.$$

Therefore we have

$$|G^{(j)}(x)| < \varphi(x) . 2^{-n} \text{ for all } j \in \{0, 1, \ldots, k\},$$

i.e. $G(x)$ satisfies 2.11, too.

Notice that 2.5. holds also for $n = 0$; then $C = \emptyset$ and $h(x) = g(x)$.

**2.6. Lemma.** Let $\varphi(x)$ be a continuous positive function. Let $n$ and $k$ be positive integers. Let $D = \{d_1, \ldots, d_n\} \subset R$, $d \notin D$ and $C$ be a dense subset of $R$. Let $c_1, \ldots, c_n \in R$ and let $F(x)$ be a continuous surjective function such that $F(c_i) = d_i$ for $i = 1, \ldots, n$. Then there is an analytic function $G(x)$ and a number

$c \in C$ such that $G(x)$ vanishes at $c_1, \ldots, c_n$, $F(c)+G(c)=d$ and (2.11) holds.

**Proof.** By 2.3. there is an analytic positive function $g(x)$ satisfying 2.4. Let $h(x)$ be then function defined by (2.12) and let $L$ be a common bound for the functions

$$\frac{h(x)}{\varphi(x)}, \ldots, \frac{h^{(k)}(x)}{\varphi(x)}. \text{ Put } p=\frac{1}{2^n . L}.$$

Since $F(x)$ is surjective, there is $y \in R$ such that $F(y)=d$. Since $d \notin D$, we have $y \neq c_i$ for $i=1, \ldots, n$ and therefore $h(y) \neq 0$.
The functions $F(x)$ and $h(x)$ are continuous at the point $y$, hence there is positive $\delta$ such that

$$|h(y)-h(t)|<\frac{|h(y)|}{2} \text{ and } |F(y)-F(t)|<\frac{p . |h(y)|}{2}$$

whenever $|y-t|<\delta$. The set $C$ is dense, hence there is

$$c \in C \cap (y-\delta, y+\delta) \text{ such that } c \neq c_i \text{ for } i=1, \ldots, n.$$

Put $r=\dfrac{d-F(c)}{h(c)}$ (obviously $h(c) \neq 0$). Since $|y-c|<\delta$, we have

$$|h(y)|-|h(c)| \leqq |h(y)-h(c)|<\frac{|h(y)|}{2},$$

therefore

$$\frac{1}{|h(c)|}<\frac{2}{|h(y)|}.$$

Further we have

$$|F(y)-F(c)|=|d-F(c)|<\frac{p . |h(y)|}{2},$$

hence $|r|<p$.

Now define

$$G(x)=r . h(x).$$

Then $F(c)+G(c)=d$. The other required properties of $G$ can be easily verified.

**2.7. Proof of Theorem 1.1.** We may obviously assume that $\varphi(x)<1$ for every real $x$. Let

$$a_1, a_2, a_3, \ldots$$
$$b_1, b_2, b_3, \ldots$$

be one to one lists of the elements of $A$, $B$, respectively. The function $f(x)$ will be presented in the form

$$f(x)=x+\sum_{n=1}^{\infty} g_n(x), \tag{2.13}$$

where the functions $g_n(x)$ will be constructed by induction. Let $n=1$. By 2.5. there is an analytic function $g_1(x)$ such that $a_1+g_1(a_1)\in B$ and

$$\max\{|g_1(x)|, \ldots, |g^{(k)}(x)|\}<\frac{\varphi(x)}{2}.$$

Denote $a_1=c_1+g_1(a_1)=d_1$, $C_1=\{c_1\}$, $D_1=\{d_1\}$. Now suppose that we have already sets $C_n=\{c_1, \ldots, c_n\}\subset A$, $D_n=\{d_1, \ldots, d_n\}\subset B$ and analytic functions $g_1(x), \ldots, g_n(x)$ such that $f_n(c_i)=d_i$ for $i=1, \ldots, n$, where

$$f_n(x)=x+\sum_{j=1}^{n} g_j(x) \tag{2.14}$$

and

$$\max\{|g_i(x), |g_i'(x)|, \ldots, |g_i^{(k)}(x)|\}<\frac{\varphi(x)}{2^i}$$

for $i=1, \ldots, n$ and for all $x\in R$.

We shall distinguish two cases: $n$ is even and $n$ is odd.

a) If $n$ is even, denote $s_n=\min\{i\in N\colon a_i\notin C_n\}$ and $a_{s_n}=c_{n+1}$.

Then $c_{n+1}\notin C_n$ and hence by 2.5. there is an analytic function $g_{n+1}(x)$ such that

$$g_{n+1}(c_i)=0 \text{ for } i=1, \ldots, n,$$

$$f_n(c_{n+1})+g_{n+1}(c_{n+1})\in B$$

and

$$\max\{|g_{n+1}(x)|, |g_{n+1}'(x)|, \ldots, |g_{n+1}^{(k)}(x)|\}<\frac{\varphi(x)}{2^{n+1}}. \tag{2.15}$$

Denote

$$f_n(c_{n+1})+g_{n+1}(c_{n+1})=d_{n+1},\ C_n\cup\{c_{n+1}\}=C_{n+1},$$

$$D_n\cup\{d_{n+1}\}=D_{n+1},\quad f_n(x)+g_{n+1}(x)=f_{n+1}(x).$$

b) If $n$ is odd, denote

$r_n=\min\{i\in N\colon b_i\notin D_n\}$ and $b_{r_n}=d_{n+1}$; obviously $d_{n+1}\notin D_n$.

Since the functions $g_1(x)$, ..., $g_n(x)$ are bounded and continous, the function $f_n(x) = x + \sum_{i=1}^{n} g_i(x)$ is surjective.

Therefore by Lemma 2.6. there is an analytic function $g_{n+1}(x)$ and a point $c_{n+1} \in A$ such that $g_{n+1}(x)$ vanishes at the points $c_1, \dots, c_n$, $f_n(c_{n+1}) + g_{n+1}(c_{n+1}) = d_{n+1}$ and it holds (2.15).

Denote $C_{n+1} = C_n \cup \{c_{n+1}\}$, $D_{n+1} = D_n \cup \{d_{n+1}\}$, $f_{n+1}(x) = f_n(x) + g_{n+1}(x)$.

Define a function $f(x)$ by

$$f(x) = x + \sum_{n=1}^{\infty} g_n(x). \tag{2.16}$$

Since $|g_n(x)| < \varphi(x).2^{-n}$, the right-hand side of (2.16) converges for all reals $x$. Since all $g_n(x)$ are analytic functions also the function $f(x)$ is analytic.

Since $|g'_n(x)| < \varphi(x).2^{-n} < 2^{-n}$, we have

$$f'(x) = 1 + \sum_{n=1}^{\infty} g'_n(x) > 0,$$

i.e. the function $f(x)$ is increasing. It is easy to see that $f(x)$ satisfies the condition (1.1).

It is also easy to verify that $A = \bigcup_{n=1}^{\infty} C_n$ and $B = \bigcup_{n=1}^{\infty} D_n$. Since $g_m(c_j) = 0$ for all $m > j$, we have $f(C_n) = D_n$ and therefore $f(A) = B$, too.

**2.8. Proof** of **Theorem 1.2.** Since the proof is very similar to that of Theorem 1.1. we shall only schetch it. Let $A = \bigcup_{i \in N} A_i$, $B = \bigcup_{i \in N} B_i$. The functions $g_n$ will be constructed in the same way as in the previous proof. However, if $n$ is even and $c_{n+1} \in A_i$ then instead of

$$f_n(c_{n+1}) + g_{n+1}(c_{n+1}) \in B - D_n,$$

we have to arrange

$$f_n(c_{n+1}) + g_{n+1}(c_{n+1}) \in B_i - D_n.$$

Analogously, if $n$ is odd and $d_{n+1} \in B_i$, then instead of $c_{n+1} \in A$ we have to arrange $c_{n+1} \in A_i$.

## 3. Remarks

We point out again that although we can choose the function in 1.1. so that it is analytic, we can never more result in a polynomial.

**3.1. Proposition.** There are two countable dense sets $A$ and $B$ in $R$ such that $P(A) \neq B$ for every polynomial $P$.

**Proof.** Let $P_1$, $P_2$, $P_3$, ... be a one to sequence of all polynomials with rational coefficients.

Put $I = \{i \in N: P_i(Q) \cap (i, i+1) \neq \emptyset\}$, where $Q$ is the set of rational numbers.

For each $i \in I$ let us choose an arbitrary element

$$r_i \in P_i(Q) \cap (i, i+1).$$

Let us denote $A = Q$, $B = Q - \{r_i: i \in I\}$.

Let $P$ be a polynomial such that $P(A) = B$. Since $P(A)$ contains only rational numbers all the coefficients of $P$ are rational. Therefore $P = P_i$ for some $i \in N$. However, then we have $r_i \in P_i(A)$, $r_i \notin B$ which contradicts $P(A) = B$.

**3.2. Proposition.** Let $K$ be the set of all complex numbers. Let $f(x)$ be a complex analytic function such that

$$|f(x) - x| < e^{-|x|} \quad \text{for each} \quad x \in K.$$

Then $f$ is the identical function.

**Proof.** The assertion follows from the Liouville theorem.

This assertion shows that it is not possible to extend Theorem 1.1. to complex numbers.

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## SÚHRN

### MONOTÓNNE ZOBRAZENIA MEDZI HUSTÝMI SPOČÍTATEĽNÝMI MNOŽINAMI REÁLNYCH ČÍSEL

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V práci sa dokazuje, že pre ľubovoľné dve husté spočítateľné podmnožiny reálnej osi, ľubovoľnú kladnú spojitú funkciu $\varphi(x)$ a ľubovoľné prirodzené číslo $n$ existuje analytická reálna funkcia $f(x)$ taká, že $f(A) = B$ a pre všetky reálne čísla $x$ platí

$$\max(\{|f(x) - x|, |f'(x) - 1|, |f''(x)|, \ldots, |f^{(n)}(x)|\}) < \varphi(x).$$

Toto tvrdenie sa zovšeobecňuje na prípad spočítateľných systémov $\{A_i\colon i \in N\}$, $\{B_i\colon i \in N\}$ po dvoch disjunktných hustých spočítateľných množín; vtedy $f(A_i) = B_i$ pre všetky $i \in N$.

Dokazuje sa tiež, že v uvedenom tvrdení nemožno nahradiť reálnu os rovinou komplexných čísel, ani analytickú funkciu $f(x)$ polynómom.

## РЕЗЮМЕ

### ОТОБРАЖЕНИЯ МЕЖДУ СЧЕТНЫМИ ПЛОТНЫМИ ПОДМНОЖЕСТВАМИ ВЕЩЕСТВЕННОЙ ПРЯМОЙ, СОХРАНЯЮЩИЕ ПОРЯДОК

Ян Борсик, Кошице, Иван Корец, Братислава

В работе доказывается следующая теорема:

Пусть $A$ и $B$ — счетные плотные подмножества вещественной прямой, пусть $\varphi(x)$ — положительная вещественная непрерывная функция вещественного переменного и $n$ — натуральное число. Тогда существует аналитическая функция $f(x)$ такая, что $f(A) = B$ и для всех $x$

$$\max(\{|f(x) - x|, |f'(x) - 1|, |f''(x)|, \ldots, |f^{(n)}(x)|\}) < \varphi(x)$$

Эта теорема обобщается для случая счетных систем $\{A_i; i \in N\}$, $\{B_i, i \in N\}$ попарно непересекающихся счетных плотных подмножеств вещественной прямой; тогда $f(A_i) = B_i$ для всех $i \in N$.

Далее доказывается, что теорема не будет верной, если в ней заменить вещественную прямую комплексной плоскостью или аналитические функции полиномами.