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ON WEAK FORMS OF CONTINUITY OF FUNCTIONS AND MULTIFUNCTIONS

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The notion of the weak continuity of a function was introduced by Levine [5] and studied by several other authors. To many generalized forms of the continuity one can associate weak variants. If the range of a function is a regular topological space, then any of such weak continuity notions coincides with the original one. The present paper discusses the situation in the case of a quasiregular topological space. On the other hand also the multifunctions are considered and it is shown that the situation in this case may be different. In connection with the mentioned study a characterization of quasiregular and normal topological spaces may be obtained.

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If nothing else is said, X and Y denote arbitrary topological spaces. Besides of mappings $f\colon X\to Y$, we consider also so called multivalued mappings $F\colon X\to P(Y)$ where $P(Y)$ is the power set of Y. For the case of simplicity we use the notation $F\colon X\to Y$ for the multivalued mapping to $P(Y)$. We suppose $F(x)\neq\emptyset$ for any $x\in X$. The notation $\bar{A}$ stands for the closure of a set A, int A denotes the interior of A.

Definition 1. ([5]). A mapping $f\colon X\to Y$ is said to be weakly continuous at a point $x\in X$ if for any neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(U)\subset\bar{V}$. It is said to be weakly continuous, if it is weakly continuous at any $x\in X$.

In a quite natural way the notion of weak quasicontinuity may be used, to generalize the notion of the quasicontinuity ([2]). The notion of weak quasicontinuity has been used e.g. in [8].

Definition 2. A mapping $f\colon X\to Y$ is said to be weakly quasicontinuous at a point $x\in X$, if for any neighbourhood U of x and any neighbourhood V of $f(x)$ there exists a nonempty open set $G\subset U$ such that $f(G)\subset\bar{V}$. It is said to be weakly quasicontinuous ($w.q.$) if it is weakly quasicontinuous at any $x\in X$.

It is obvious that the quasicontinuity (where $f(G) \subset \bar{V}$ is substituted by $f(G) \subset V$) implies the weak quasicontinuity and both of them coincide if Y is a regular space.

In the study of quasicontinuous functions, the notion of somewhat continuity is usefull (see [1]). We recall the definition and simultaneously we present the definition of the weak somewhat continuity.

Definition 3. A function $f: X \to Y$ is called somewhat continous (weakly somewhat continuous) if for any open $V \subset Y$ for which $f^{-1}(V) \neq \emptyset$ we have int $f^{-1}(V) \neq \emptyset$ (int $f^{-1}(\bar{V}) \neq \emptyset$).

Again the somewhat continuity (s.c.) implies the weak somewhat continuity (w.s.c.) and both of them coincide if Y is a regular space.

The relation between w.q. and w.s.c. may be described in an analogical way as it is described between quasicontinuity and somewhat continuity ([6]). We present this relation in the following proposition, omitting its easy proof.

Proposition 1. A funkction $f: X \to Y$ is w.q. if and only if there exists a basis $\mathcal{B}$ of open sets in X such that for any $B \in \mathcal{B}$ f/B is w.s.c.

The notion of a quasiregular topological space is a slight generalization of a regular topological space. It seems to be of interest how are related the generalized continuities and their weak forms in quasiregular spaces.

Definition 4. A topological space Y is said to be quasiregular if for any nonempty open set $G \subset Y$ there exists a nonempty open set $U \subset G$ such that $\bar{U} \subset G$.

The following example shows that in case of a quasiregular space Y, there may exist a function $f: X \to Y$ such that f is weakly quasicontinuous and not somewhat continuous. Hence the example simultaneously shows that weak quasicontinuity does not imply quasicontinuity and weak somewhat continuity does not imply somewhat continuity if the space of values is quasiregular.

Example 1. Let $X = \langle 0, 1 \rangle$ with the usual topology. Put $Y = \langle 0, 1 \rangle$ where the topology of Y consists of the usual topology and moreover of all the sets $G - \left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$ where G is open in the usual topology. The space Y is not regular (see e.g. [3], p. 58) but one can see that it is a quasiregular space.

Let $f: X \to Y$ be defined as $f(0) = 0$ and

$$f(x) = \frac{1}{n}, \text{ if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right\rangle; \qquad n = 1, 2$$

The function f is not somewhat continuous, because if $V = \langle 0, 1 \rangle - \left\{1, \frac{1}{2}, \ldots\right\}$ then $f^{-1}(V) = \{0\} \neq \emptyset$, while int $f^{-1}(V) = \emptyset$. But for any open $V \subset Y$ and any open

$G \subset X$ for which $f^{-1}(V) \cap G \neq \emptyset$ we have $\operatorname{int} f^{-1}(\bar{V}) \cap G \neq \emptyset$. So, by Proposition 1, f is weakly quasicontinuous.

The following proposition gives a condition for the equivalence of the weak somewhat continuity and somewhat continuity for function with values in a quasiregular topological space.

Proposition 2. Let Y be a quasiregular space and $f\colon X \to Y$. Let $f(X)$ be dense in Y. Then f is somewhat continuous if and only if it is weakly somewhat continuous.

Proof. It is sufficient to prove that the weak somewhat continuity implies the somewhat continuity. The converse is obvious. Suppose $f^{-1}(V) \neq \emptyset$ for an open set $V \subset Y$. Then $V \neq \emptyset$ and by the quasiregularity of Y there exists a nonempty open set $G \subset V$ such that $\bar{G} \subset V$. Since $f(X)$ is dense in Y, we have $f(X) \cap G \neq \emptyset$, hence $f^{-1}(G) \neq \emptyset$. The weak somewhat continuity implies $\operatorname{int} f^{-1}(\bar{G}) \neq \emptyset$, hence $\operatorname{int} f^{-1}(V) \neq \emptyset$.

Corollary. Let Y be a quasiregular topological space. Then any mapping $f\colon X \to Y$, onto Y, is somewhat continuous if and only if it is weakly somewhat continuous.

The last Corollary can not be proved if we substitute somewhat continuity by quasicontinuity and weak somewhat continuity by weak quasicontinuity. Namely we can not prove that the weak quasicontinuity implies quasicontinuity neither in the case when f is onto Y.

Example. Let $X = \langle 0, 2 \rangle$ with the usual topology. Let Y have the same meaning as in Example 1.

Put

$$f(0) = 0,$$

$$f(x) = \frac{1}{n}, \text{ if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right\rangle; \qquad n = 1, 2, \ldots,$$

$$f(x) = x - 1, \text{ if } x \in (1, 2\rangle.$$

Then $f\colon X \to Y$ is onto Y, weakly quasicontinuous, but it is not quasicontinuous at $x = 0$.

The equivalence of weak somewhat continuity and somewhat continuity of certain mappings gives a characterization of quasiregular spaces. It may be formulated by means of bijective mappings.

Theorem 1. A topological space Y is quasiregular if and only if for any topological space X the following holds. If $f\colon X \to Y$ is one-to-one onto Y, then f is weakly somewhat continuous if and only if it is somewhat continuous.

Proof. Let Y be quasiregular and $f\colon X \to Y$ a bijection. The fact that f is weakly somewhat continuous if and only if it is somewhat continuous is guaranteed by Corollary of Proposition 2.

Now let Y be not quasiregular. We have to construct a topological space X and a bijective mapping $f\colon X \to Y$ such that f is weakly somewhat continuous but not somewhat continuous. Since Y is not quasiregular there exists nonempty open set G in Y such that for any nonempty open $H \subset G$ we have $\bar{H} \cap (Y - G) \neq \emptyset$. Put $X = Y$ and define a topology on X as follows. A set $A \subset X$ will be open in X if and only if $A = X$ or $A \subset (Y - G)$. To avoid difficulties in what follows we denote int $_x E$ the interior in the space X of a set E and $\bar{E}_y$ the closure (in the space Y) of a set E.

Let $f\colon X \to Y$ be the identity mapping. Let V be an open set in Y such that $f^{-1}(V) \neq \emptyset$. Since $f^{-1}(V) = V$ we have $V \neq \emptyset$. If $V \subset G$, we have $\bar{V}_y \cap (X - G) \neq \emptyset$. Hence the set $\bar{V}_y \cap (X - G)$ is open in X.

So we have

$$\operatorname{int}_x f^{-1}(\bar{V}_y) \supset \operatorname{int}_x f^{-1}(\bar{V}_y \cap (X - G)) = \operatorname{int}_x (\bar{V}_y \cap (X - G)) =$$
$$= \bar{V}_y \cap (X - G) \neq \emptyset.$$

In what follows we define $G' = Y - G$.

If $V \not\subset G$, then $V \cap G' \neq \emptyset$, hence $V \cap G'$ is open in X. So

$$\operatorname{int}_x f^{-1}(\bar{V}_y) \supset \operatorname{int}_x f^{-1}(V) \supset \operatorname{int}_x f^{-1}(V \cap G') = V \cap G' \neq \emptyset.$$

In both the cases $\operatorname{int}_x f^{-1}(\bar{V}_y) \neq \emptyset$, hence f is weakly somewhat continuous.

Taking $V = G$, we have $f^{-1}(V) \neq \emptyset$, but

$$\operatorname{int}_x f^{-1}(V) = \operatorname{int}_x f^{-1}(G) = \emptyset.$$

Hence f is not somewhat continuous.

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Considering mapping $f\colon X \to Y$, the regularity of Y is always sufficient for equivalence of the mentioned type of generalized continuity to its weak form. We show that it is not the case if we consider multifunctions. First recall the notions.

The continuity of a multifunction $F\colon X \to Y$ is defined by means of its upper and lower continuity (see [4] p. 405) Its generalization to the weak continuity is straightforward. We restrict our attention to the definitions of the generalized continuity notions.

Definition 5. (See [7] [8]). A multifunction $F\colon X \to Y$ is called upper semi-quasicontinuous at $x_0 \in X$ if for any open set V containing $F(x_0)$ and for any open set U containing x_0 there exists a nonempty open $G \subset U$ such that $F(x) \subset V$ for any $x \in G$.

It is said to be lower semi-quasicontinuous at $x_0 \in X$, if for any open V for which $V \cap F(x_0) \neq \emptyset$ and for any open U containing x_0 there exists a nonempty open set $G \subset U$ such that $F(x) \cap V \neq \emptyset$ for any $x \in G$.

Note that we can immediately obtain the corresponding weak notions substituing in Definition 5 the relations $F(x)\subset V$ or $F(x)\cap V\neq\emptyset$ by $F(x)\subset\bar{V}$ or $F(x)\cap\bar{V}\neq\emptyset$ respectively.

The upper semi-quasicontinuity of F means the upper semi-quasicontinuity in every point $x\in X$. It is similar in the case of the lower semi-quasicontinuity.

The notion of the upper (lower) inverse image $F^{+}(A)$ ($F^{-}(A)$ is frequently used. It is defined for $A\subset Y$ as

$$F^{+}(A)=\{x\colon x\in X,\ F(x)\subset A\},\ F^{-}(A)=\{x\colon x\in X,\ F(x)\cap A\neq\emptyset\}.$$

By means of F^{+} and F^{-} one can define upper and lower somewhat continuity (see [7]).

Definition 6. A multifunction $F\colon X\to Y$ is called upper (lower) somewhat continuous if for any open set $V\subset Y$ for which $F^{+}(V)\neq\emptyset$) we have int $F^{+}(V)\neq\emptyset$ (int $F^{-}(V)\neq\emptyset$).

The definition of the corresponding notions of weakly upper (lower) somewhat continuities is obtained substituing int $F^{+}(V)\neq\emptyset$ (int $F^{-}(V)\neq\emptyset$) by int $F^{+}(\bar{V})\neq\emptyset$ (int $F^{-}(\bar{V})\neq\emptyset$).

We do not discuss the analogies of the results obtained in the first part of the paper for multifunctions. We restrict our attention to the fact that the equivalence of the types of generalized continuities to their weak forms in case of multifunctions depends on normality of the space of values. In fact we can give the following characterization of normal spaces.

Theorem 2. A topological space Y is normal if and only if any of the following conditions is satisfied for any topological space X.

(i) If $F\colon X\to Y$ is a closed valued multifunction, then F is upper somewhat continuous if and only if it is weakly upper somewhat continuous.

(ii) If $F\colon X\to Y$ is a closed valued multifunction, then F is upper semi-quasicontinuous if and only if it is weakly upper semi-quasicontinuous.

Proof. The proofs of necessity of either of the conditions (i), (ii) are straightforward and similar. So we prove the necessity of (i) only.

Let Y be normal and $F\colon X\to Y$ weakly upper somewhat continuous and closed valued multifunction. Let V be open in Y such that $F^{+}(V)\neq\emptyset$. Then $F(x)=V$ for some $x\in X$. Since $F(x)$ is closed and Y normal there exists an open set G such that

$$F(x)\subset G\subset\bar{G}\subset V.$$

Hence $F^{+}(G)\neq\emptyset$ and by the weak upper somewhat continuity int $F^{+}(\bar{G})\neq\emptyset$. The last implies int $F^{+}(V)\neq\emptyset$, proving that F is upper somewhat continuous.

To prove the sufficiency of (i), (ii) it suffices to prove the following. If Y is not a normal space then there exists a topological space X and a closed valued multifunction $F\colon X\to Y$ such that F is weakly upper semi-quasicontinuous but not upper somewhat continuous. (It follows from the obvious relations between the upper semi-quasicontinuity and upper somewhat continuity and from analogical relations between their weak variants.)

So let Y be not a normal space. Then there exists a closed set $F \subset Y$ and an open set $G \supset F$ such that for any open set $H \supset F$, we have $\bar{H} \cap (Y - G) \neq \emptyset$.

Put $X = Y$ and define the topology $\mathcal{T}$ on X in the same way as in Theorem 1, i. e. $A \in \mathcal{T}$ if and only if $A \subset X - G$ or $A = X$. Define the multifunction F: $X \to Y$ such that

$$F(x) = \{x\} \cup F, \text{ for any } x \in X.$$

Then F is closed valued. We prove that it is weakly upper semi-quasicontinuous at any $x_0 \in X$. Let $x_0 \in G$. Take any open set V in Y containing $F(x_0) = \{x_0\} \cup F$ and any open set U in X containing x_0. Since U contains $x_0 \in G$, we have from the definition of $\mathcal{T}$ that $U = X$. So to prove weak upper semiquasicontinuity at x_0 we have only to find an open set W in X such that $W \neq \emptyset$ and $F(x) \subset \bar{V}_y$ for any $x \in W$. Put $W = \bar{V}_y \cap (X - G)$. Evidently $W \neq \emptyset$ and hence W is as nonempty open subset of the space X. Now for any $x \in W$ we have

$$F(x) = \{x_0\} \cup F \subset W \cup F \subset W \cup \bar{V}_y \subset \bar{V}_y$$

Thus the weak upper semi-quasicontinuity at $x_0 \in G$ is proved.

If $x_0 \notin G$ then the proof is trivial because x_0 is any isolated point in the topology of the space X. Thus weak upper semi-quasicontinuity of F is proved.

The function F is not somewhat continuous. In fact if we put $V = G$, we have a nonempty open set in Y for which $F^+(V) = V \neq 0$. But $\text{int}_x F^+(V) = \text{int}_x G = \emptyset$.

Remark. It may be easily seen that if we consider a lower semi-quasicontinuity and weak lower semi-quasicontinuity of multifunctions, then regularity of the space of values is sufficient for their equivalence. The same is true for a lower somewhat continuity and weak lower somewhat continuity.

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SÚHRN

O SLABÝCH FORMÁCH SPOJITOSTI FUNKCIÍ A MULTIFUNKCIÍ

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Skúmajú sa vzťahy rôznych foriem spojitosti funkcií a multifunkcií a k nim zodpovedajúcich slabých foriem. Dôraz sa kladie na funkcie nadobúdajúce hodnoty v kvaziregulárnych priestoroch. V tejto súvislosti sa podáva tiež istá charakterizácia kvaziregulárnych priestorov.

РЕЗЮМЕ

О СЛАБЫХ ВИДАХ ОТОБРАЖЕНИЙ И МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ

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Исследуются соотношения между различными типами непрерывности и их слабыми видами. Обычно изучаются отображения принимающие значения в квазирегулярных пространствах. В связи с тем характеризуются квазирегулярные пространства.