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ON COMPATIBILITY IN QUANTUM LOGICS

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Introduction

The notion of compatibility in quantum logics is an important tool in this theory. It is a useful notion from the mathematical point of view and on the other hand it has its own physical meaning. Many modifications of this notion are known in the literature. The aim of this paper is to compare various of them and to illustrate their applications in two directions. The first is the problem of existence of a Boolean σ-algebra containing a subset of a given logic L and contained in L. The second one is the problem of existence of joint observables and joint distributions. The results we are dealing with have been mostly published. What is new are some interrelations between them. Only some of the proofs giving the interrelations or explaining the basic ideas are given. As to the others we give the corresponding refferences. To make the paper self contained, at least in certain extent, we give some well-known examples in the preliminary part.

Preliminaries

By the mathematical description of quantum experiments, a generalization of classical probability theory is needed. While in the classical probability theory the set of all "experimentally verifiable propositions" of the physical system (or, equivalently, the set of all random events) can be mathematically described as a Boolean σ-algebra, in the quantum case a more general algebraic structure is needed. The reason for this is the fact, that there exist pairs of physical quantities (e.g. position and momentum of a particle) which cannot be measured simultaneously with an arbitrary accuracy — as it can be seen by the well-known Heisenberg uncertainty principle. In the quantum logic approach to quantum mechanics, the basic concepts are the set L of all experimentally verifiable propositions of the physical system and the set M of physical states. L is usually suposed to be a partially ordered set with the greatest element 1 and the least element 0, with the orthocomplementation $\perp : L \to L$ such that

(i) $(a^{\perp})^{\perp} = a, \; a \in L$

(ii) $a \leqq b$ if and only if $b^{\perp} \leqslant a^{\perp}, \; a, \; b \in L$

(iii) $a \vee a^{\perp} = 1$ for all $a \in L$

(where we denote by $x \wedge y$, resp. $x \vee y$ the infimum, resp. supremum of x and y of L if they exist); and with the orthomodular property

(iv) $a \leqslant b (a, \; b \in L)$ implies $\exists c \in L; \; c \leqslant a^{\perp}$ and $b = a \vee c$;

and which is closed under the formations of the suprema $\vee a_i$ for any sequences $\{a_i\} \subset L$ such that $a_i \leqslant a_j^{\perp}, \; i \neq j$. A set L with the properties just described is called a logic.

The elements $a, \; b \in L$ are called orthogonal ($a \perp b$) if $a \leqslant b^{\perp}$; and they are compatible ($a \leftrightarrow b$) if there are elements $a_1, \; b_1, \; c$ in L mutually orthogonal and such that $a = a_1 \vee c$ and $b = b_1 \vee c$.

Let us mention some important examples of logics.

Example 1. Let H be a separable Hilbert space and $L(H)$ the collection of all closed subspaces of H. If the orthocomplementation is defined as the usual orthocomplementation in H and the ordering is given by inclusion, we obtain a logic.

Example 2. Let Ω be a set and S a nonempty collection of subsets of Ω closed under complements and countable disjoint unions. If the ordering is defined by inclusion and if for $E \in S$ we define $E^{\perp}$ as the set-theoretical complement, we have another example of a logic. This logic is known as σ-class [5]. It is also called a $q - \sigma$-algebra [11], or a G-logic [17].

Remark 1. The last example is a straightforward generalization of Boolean σ-algebra of sets. There are σ-classes which are not σ-algebras. A simple but useful σ-class for giving various counterexamples is the following (see [16]).

Example 3. $\Omega = \{1, 2, \ldots, 8\}$ and S is the collection of all subsets of Ω with even number of elements.

Remark 2. Note that two elements $E, \; F$ of a σ-class S are compatible in S exactly if $E \cap F \in S$.

A state on the logic L is a probability measure on L, i.e. a map $m\colon L \rightarrow [0, 1]$ such that

(i) $m(1) = 1$

(ii) $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$ for any sequence $\{a_i\}$ of mutually orthogonal elements of L. A set of states M is called full for L if $m(a) \leqslant m(b)$ for all $m \in M$ implies $a \leqslant b$ $(a, \; b \in L)$.

Let L_1 and L_2 be two logics. The mapping h from L_1 into L_2 is called a σ-homomorphism if

(i) $h(1_1) = 1_2$ (where 1_1 and 1_2 are the greatest elements in L_1 and L_2, respectively),

(ii) $p \perp q$, $p, q \in L_1$ implies $h(p) \perp h(q)$,

(iii) $h(\bigvee p_i) = \bigvee h(p_i)$ for any sequence $\{p_i\}$ of mutually orthogonal elements of L_1.

With the help of the concept of σ-homomorphism we introduce observables (corresponding to physical quantities). If R is the real line and $B(R)$ is the σ-algebra of all Borel sets, then any σ-homomorphism of $B(R)$ into L is called an observable on L. If x is an observable and $f: R \to R$ is a Borel measurable function, then the map $f(x): E \mapsto x(f^{-1}(E))$ is also an observable, which is called the function f of the observable x. Spectrum $\sigma(x)$ of the observable x is the smallest closed set $C \subset R$ such that $x(C) = 1$. An observable x is bounded if its spectrum $\sigma(x)$ is a compact set. If x is an observable and m is a state on L, then the map $m_x: E \mapsto m(x(E))$ is a probability measure on $B(R)$. It is called the probability distribution of the observable x. The expectation of the observable x in the state m can be defined by $m(x) = \int_R t m_x(dt)$, if the integral exists.

A subset L_0 of a logic L is called a sublogic of L if

(i) $a \in L_0$ implies $a^\perp \in L_0$

(ii) $a_1, a_2, \ldots \in L_0$ with $a_i \perp a_j$ implies $\bigvee a_i \in L_0$.

It can be easily verified that a sublogic of a logic is a logic itself.

A sublogic $L_0 \subset L$ is called a lattice sublogic provided that $a, b \in L_0$ implies $a \vee b$ exists in L and is in L_0. In this case L_0 is a lattice.

If a lattice sublogic is distributive, then it is a Boolean σ-algebra and it is called a sub-σ-algebra of L.

Note that some authors define the sublogic of a logic by means of the notion of the isomorphism of the following kind.

If L_1 and L_2 are two logics, then an isomorphism from L_1 into L_2 is defined as an injection $h: L_1 \to L_2$ such that

(i) $a, b \in L_1$ and $a \perp b$ implies $h(a) \perp h(b)$,

(ii) a_n $(n = 1, 2, \ldots) \in L_1$ and $\bigvee a_n$ exists in L_1 implies that there exists $\bigvee h(a_n)$ in L_2 and $h(\bigvee a_n) = \bigvee h(a_n)$,

(iii) $h(0_1) = 0_2$, $h(1_1) = 1_2$, where 0_i, 1_i $(i = 1, 2)$ are the least and the greatest elements in L_i, respectively.

A logic $L_0 \subset L$ is then defined to be a sublogic of L if the identity map from L_0 into L is an isomorphism.

In what follows the sublogic will be understood not in this sense, but in the sense of the first definition. If we consider the sublogic in the sense of the isomorphism we say it explicitly.

Note that the two definitions are not equivalent. To see this (see [21]) it suffices to consider the following example.

Example 4. Let $\Omega = \{1, 2, ..., 8\}$ and let S be the σ-class of all subsets of Ω. Let S_1 be the σ-class consisting of all subsets of Ω with even number of elements. Then S_1 is a sublogic of S. But it is not a sublogic defined in the sense of isomorphism. If fact if we take $E = \{3, 4, ..., 8\}$, $F = \{2, 4, ..., 8\}$, we have $E \vee F = \{1, 2, ..., 8\}$ in S_1, but $h(E) \vee h(F) = \{2, 3, ..., 8\}$ in the logic S.

Of course, it is caused by a rather strong definition of the isomorphism. Nevertheless, sometimes it has certain reason to consider the notion of a sublogic in the last sense.

Note that in case of sub-σ-algebra there are no difficulties, even if we consider the definition by means of an isomorphism. It can be easily seen that such definition gives the same result.

Compatibility and subalgebras of logics

Compatible sets are important in the problem of existence of Boolean σ-subalgebras. A set $A \subset L$ is said to be compatible if $a \leftrightarrow b$ for any two elements $a, b \in A$.

Varadarajan [22] proved the following statement.

Theorem 1. Let L be a logic which is simultaneously a lattice. Then a necessary and sufficient condition for the existence of a Boolean sub-σ-algebra B such that $A \subset B \subset L$ is that A is a compatible (in L) set.

The theorem was formulated in [22] for logics which are not necessarily lattices. But it was shown by Pool [16] and independently by Ramsay [19] that it is not true in such general formulation.

Example 5. Let $\Omega = \{1, 2, ..., 8\}$ and let S_1 have the same meaning as in Example 4. The subclass $A = \{E, F, G\}$ where $E = \{1, 2, 3, 4\}$, $F = \{1, 2, 5, 6\}$, $G = \{1, 3, 6, 8\}$ is compatible (in S_1) but there is not a Boolean σ-algebra B of sets such that $A \subset B \subset S_1$. In fact, if such a σ-algebra existed, then it would contain $(E \cup F) \cap G$. But the last is not true.

One can see in the above example that the set E is not compatible with $F \cup G$ in spite of the mutual compatibility of E, F, G. So the following condition seems to be important for a logic L.

Let L be a logic. We say that L satisfies the condition (c) if for any three elements a, b, c which are mutually compatible, we have

$$a \leftrightarrow b \vee c \tag{c}$$

The following theorem is true.

Theorem 2. Let L be a logic satisfying (c). Then to any compatible set $A \subset L$ there exists a Boolean sub-σ-algebra B such that $A \subset B \subset L$.

The proof of this theorem for σ-classes may be found in [5]. (As to the idea of the proof and also the idea of the proofs which follow, it will be given later). It is proved for general logics in [21] and in [10]. In the paper [21] there was given also a condition when a generated sub-σ-logic in the sense of the isomorphism coincide with the generated Boolean sub-σ-algebra. There is proved the following theorem.

Theorem 3. If a σ-logic L satisfies the condition (c) and $A \subset L$ is compatible, then there exists a Boolean sub-σ-algebra B of L such that $A \subset B \subset L$, Moreover, the σ-sublogic (in the sense of isomorphism) generated by A coincides with the σ-algebra generated by A.

Note that the σ-sub-logic generated by a compatible set in general does not coincide with the σ-algebra generated by this set. One can take Example 4 and consider S_1 for A. Here S_1 is the generated sub-σ-logic, which is not *a sub-σ-algebra*. We shall mention later the conditions for the equality.

Another condition strenghthening compatibility which was used to prove the existence of a sub-σ-algebra was given in [7] and independently in [12]. We shall call it strong compatibility (s-compatibility).

Given a set $A \subset L$ the smallest logic L_0 containing A which is a sublogic of L always exists. The set A is said to be strongly compatible if any two elements $a, b \in A_0$ are compatible in L_0 (notation $a \overset{L_0}{\leftrightarrow} b$). The compatibility $a \overset{L_0}{\leftrightarrow} b$ means that there exist elements $a_1, b_1, c \in L_0$ such that they are mutually orthogonal and $a = a_1 \vee c$, $b = b_1 \cup c$. In [12] and [7] the following theorem is proved.

Theorem 4. If a set $A \subset L$ is strongly compatible then there exists a Boolean sub-σ-algebra B such that $A \subset B \subset L$.

Moreover in [12] it was proved that if the assumption of the theorem is satisfied, then the generated (by A) sub-σ-logic coincides with the generated Boolean sub-σ-algebra. The last fact was for σ-classes shown in [13].

A stronger notion of compatibility has been introduced also in [1]. To distinguish this notion we shall call it full compatibility (shortly f-compatibility).

A finite set $\{a_1, a_2, \ldots, a_n\}$ of elements of a logic L is said to be full compatible in L if there exists a finite collection of pairwise orthogonal elements $\{e_i, 1 \leqslant i \leqslant k\}$ such that for any element a_i $(1 \leqslant i \leqslant n)$ there exists a finite subcollection $\{e_{i_j}\}$ of $\{e_i\}$ such that $a_i = \bigvee_j e_{i_j}$. A set $A \subset L$ is said to be f-compatible in L if any finite subset of A is f-compatible in L.

Using f-compatibility a result in [1] was obtained which gives immediately the following.

Theorem 5. If $A \subset L$ is f-compatible in L, then there exists a sub-σ-algebra B such that $A \subset B \subset L$.

After the theorems on sub-σ-algebras have been proved using several notions of compatibility it is not difficult to see how the last are connected each with others.

Position 1. Let L be a logic. Then

(i) Both full compatibility and strong compatibility of A in L imply compatibility of A in L. The converse is not true.

(ii) Strong compatibility of A in L implies its full compatibility. The converse is not true.

(iii) In the logics where (c) is satisfied the notions of compatibility and full compatibility are equivalent. In general they are not equivalent with strong compatibility.

Proof. (i) The fact that strong compatibility or full compatibility imply compatibility follows immediately from the corresponding definitions. Compatibility does not imply strong compatibility. As an example we can take Example 4. In that example the class S_1 is compatible in S but it is not strongly compatible, since the generated logic by the logic S_1 coincides with S_1.

The fact that compatibility does not imply full compatibility may be seen from Example 5, where A is compatible in S_1. The set A is not full compatible in S_1, since there is not an orthogonal collection in S_1 such that elements of A may be covered by some its subcollection.

(ii) Since A is strongly compatible, according to Theorem 4 there exists a sub-σ-algebra B such that $A \subset B \subset L$. Let $\{a_1, \ldots, a_n\}$ be any finite subset of A. Denote for any $a \in L$ $a^0 = a$, $a^1 = a^\perp$. Then the collection $K = \{a_1^{i_1} \wedge a_2^{i_2} \wedge \ldots a_n^{i_n}\}$ where $i_j = 0$ or $i_j = 1$ is a collection of elements belonging to B. Since the elements of K are mutually othogonal and for any a_{i_0} $(i = 1, 2, \ldots, n)$ we have $a_i = a_1^{i_1} \wedge \ldots \wedge a_{i-1}^{i_{i-1}} \wedge a_i \wedge a_{i+1}^{i_{i+1}} \wedge \ldots \wedge a_n^{i_n}$, where $i_1, \ldots, i_{i-1}, i_{i+1}, \ldots, i_n$ assume values 0 and 1, $\{a_1, \ldots, a_n\}$ is fully compatible.

Full compatibility does not imply strong compatibility. It may be seen from the same example which was used in (i) to prove that compatibility does not imply strong compatibility. In fact, the set S_1 in that example is full compatible in S.

(iii) Let A be compatible in L. If (c) is satisfied, we may use Theorem 2, to obtain a Boolean algebra B such that $A \subset B \subset L$. Then we can prove as in the proof of (ii) that A is fully compatible. A is not in general strongly compatible. It has been already proved, since in the example used in (ii) the logic satisfied the condition (c).

All the results for general logics may be of course applied to σ-classes. Nevertheless since σ-classes are of a special character, we may give a special condition for the existence of a Boolean sub-σ-algebra. Such a condition was obtained by means of the notion of n-compatibility in the paper [8]. A different proof was given in [17].

Let n be a positive integer. A collection $A \subset S$, where S is a σ-class is said to be n-compatible in S provided that for any subcollection of A containing finite number of elements $E_1, E_2, \ldots, E_n$ we have $E_1 \cap E_2 \cap \ldots \cap E_n \in S$.

Theorem 6. Let S be a σ-class, $A \subset S$. Then a Boolean algebra B of sets such that $A \subset B \subset S$ exists if and only if A is n-compatible for any n.

The proof, which was obtained independently of the preceding theorems, follows from some of them as it may be seen from (i)′ of the following proposition.

Proposition 2.

(i)′ n-compatibility for any positive integer n of a collection $A \subset S$ is equivalent to f-compatibility of A.

(ii)′ 3-compatibility of a σ-class S is equivalent with the condition (c) for S.

(iii)′ Strong compatibility of a set $A \subset S$ implies n-compatibility for any integer n.

(iv)′ If S is 3-compatible, then compatibility of $A \subset S$ is equivalent with n-compatibility for any positive integer n.

Proof. (i)′ Let $A \subset S$ be n-compatible in S. Let $E_1, E_2, \ldots, E_n$ be elements of A. Then $E_1, E_2 - E_1, \ldots, E_n - E_1 \cup E_2 \cup \ldots \cup E_{n-1}$ exist in S owing to the compatibility of any subcollection of $\{E_1, \ldots, E_n\}$. But such constructed sets are orthogonal and any of the sets E_i $(i = 1, 2, \ldots, n)$ may be obtained as union of some of them. Thus A is full compatible. Conversely, let A be full compatible. Let $E_1, E_2, \ldots, E_n \in A$. Let $\{F_j\}$ $(j = 1, \ldots, m)$ be an orthogonal collection such that any E_i is a union of some of its subcollection. Then $E_1 \cap E_2 \cap \ldots \cap E_n$ is a union of a subcollection of $\{F_j\}$. Hence $E_1 \cap E_2 \cap \ldots \cap E_n \in S$ proving the n-compatibility. (ii)′ The simple proof is given in [8]. (iii)′ It follows from (i)′ and (ii) of Proposition 1. (iv)′ follows from (ii)′, (i)′ and from (iii) of Proposition 1.

We did not give any one proof of the theorems 1—5. They are proved by various methods. Roughly speaking, the methods may be divided into three groups. One uses some form of the axiom of choice, another some kind of induction and the third, usually used for σ-classes, uses some form of Sierpiński method [20] of constructing the smallest collection with given properties, containing a given collection of sets.

As to the methods using the axiom of choice, they were used e.g. in the proofs of Theorem 4 and 5 in the papers [21], [7] and [1], respectively, but also in some others. Perhaps the proof in [1] seems to be very simple. The author simply shows that taking a maximal fully compatible set conaining a f-compatible collection, we obtain a Boolean sub-σ-algebra of the logic L.

A transfinite induction method was used e.g. in [17] for the proof of Theorem 6.

A modified Sierpiński method was used in [13]. A generalization of this method was given by B. J. Pettis in [14].

Joint observables

Considering the classical case of a measurable space (Ω, S), we knew that any two S-measurable functions f, g define a homomorphism $T (= T_{f,g})$ from $B(R^2)$ into S such that $T(R^1 \times F) = g^{-1}(F)$, $T(E \times R^1) = f^{-1}(E)$ for any $E, F \in B(R^1)$.

Since the observables are generalizations of measurable functions, the following question arise.

Given two observables x, y on a logic L, does there exist a homomorphism $z(z = Z_{x,y})$ from $B(R^2)$ into L such that

$$z(E \times R^1) = x(E)$$
$$z(R^1 \times F) = y(F)$$

for any $E, F \in B(R^1)$.

If such a homomorphism exists we call it according to Gudder [6] joint observable of x, y.

It was proved by Gudder that in case the logic is a lattice, then compatibility of x, y is a necessary and sufficient condition for existence of joint observable. (The result holds for any finite number of variables and may be generalized also for infinite number of variables).

Denote $\prod_{\lambda \in D} R_\lambda$ the cartesian product of sets $R_\lambda = R^1$ for $\lambda \in D$. Let B be the σ-algebra generated by the sets $\{\pi_\lambda^{-1}(E)\}$ where $\lambda \in D$ and $E \in B(R_\lambda)$ is any Borel set. (Here π denotes the projection of $\prod_{\lambda \in D} R_\lambda$ on R). A collection $\{x_\lambda\}$ $(\lambda \in D)$ of observables is said to have a joint observable if there exists a homomorphism h from B into L such that $h(\pi_\lambda^{-1}(E)) = x_\lambda(E)$ for any $E \in B(R_\lambda)$.

It is well known that the joint observable need not exist even in the case when $\{x_\lambda\}$ $(\lambda \in D)$ are mutually compatible, if L si not a lattice. It is sufficient to take the following

Example 6. Let (Ω, S_1) be the σ-class from Example 5. Take the collection $A = \{E, F, G\}$ from that example. Define the following three observables x_1, x_2, x_3:

$$x_1(A) = \chi_E^{-1}(A),\ x_2(A) = \chi_F^{-1}(A),\ x_3 = \chi_G^{-1}(A),$$

for any $A \in B(R^1)$. (χ_C denotes the indicator of C.)

The compatibility of x_i $(i = 1, 2, 3)$ follows from the compatibility of the sets E, F, G. In spite of it the joint observable of x_1, x_2, x_3 does not exist, because the existence of such observable z implies that $z(B(R^3)) \subset L$ is a Boolean σ-algebra for which $z(B(R^3)) \supset \{E, F, G\}$. As we know this is not possible.

The existence of a joint observable for a collection $\{x_\lambda\}$ $(\lambda \in D)$ of observables may be proved if we suppose some of our stronger notions of compatibility. Then it may be proved for a general logic.

Note that the stronger notion of compatibility is defined in a natural way as the corresponding compatibility of the collection $\bigcup_{\lambda \in D} \{x_\lambda(E) : E \in B(R^1)\}$.

Then we can prove the following

Theorem 7. Let L be a logic and $K=\{x_1, x_2, \ldots, x_n\}$ be a collection of observables. Then any of the conditions

(i) K is strongly compatible

(ii) K is fully compatible

(iii) K is compatible and the logic satisfies (c)

(iv) L is a σ-class and K is n-compatible for any $n=1, 2, \ldots$

implies the existence of a joint observable for K.

According to Theorems 2—6 to prove Theorem 7 it is sufficient to prove the following.

Theorem 8. If for a collection $K=\{x_1, \ldots, x_n\}$ of observables there exists a Boolean sub-σ-algebra B of the logic L such that $B \supset \bigcup_{i=1}^{n}\{x_i(E): E \in B(R^1)\}$, then the joint observable z for K exists.

Lemma. (See [22]). Let S be a Boolean algebra of subsets of a set X and h an homomorphism from S onto a Boolean σ-algebra A. Then for any homomorphism x defined on $B(R^1)$ with the values in A there exists a real S-measurable function f such that $x(E)=h(f^{-1}(E))$ for any $E \in B(R^1)$.

Proof of **Theorem 8.** Let B be the sub-σ-algebra containing $\bigcup_{i=1}^{n}\{x_i(E): E \in B(R^1)\}$. According to the Loomis theorem [9] the σ-algebra B is a homomorphic image of a σ-algebra of subset of X under a homomorphism h. Then from Lemma we have for any x_i an S-measurable function f_i such that

$$x_i(E)=h(f_i^{-1}(E)) \text{ for any } E \in B(R^1) \tag{1}$$

Now let φ be the mapping from X into R^n such that for any $t \in X$,

$$\varphi(t)=(y_1, \ldots, y_n)$$

where $y_1=f_1(t), \ldots, y_n=f_n(t)$. Now define the homomorphism z from $B(R^n)$ into L in the following way

$$z(A)=h(\varphi^{-1}(A)), \quad A \in B(R^n) \tag{2}$$

Then for any rectangle set $A=E_1 \times E_2 \times \ldots \times E_n$ we have

$$\begin{aligned}\varphi^{-1}(A)&=\{t: (f_1(t), \ldots, f_n(t)) \in E_1 \times E_2 \times \ldots \times E_n\}\\ &=f_1^{-1}(E_1) \cap \ldots \cap f_n^{-1}(E_n),\end{aligned}$$

so that

$$\begin{aligned}z(A)=h(\varphi^{-1}(A))&=h(f_1^{-1}(E_1)) \wedge \ldots \wedge h(f_n^{-1}(E_n))=\\ &=x_1(E_1) \wedge \ldots x_n(E_n).\end{aligned}$$

For example, if $A = E_1 \times R^1 \times \dots \times R^1$, we get $z(A) = x_1(A)$.

Now if g is any real valued Borel function on R^n, then the map

$$g \circ (x_1, \dots, x_n): E \mapsto z(g^{-1}(E)), \quad E \in B(R^1),$$

is an observable, the range of which is contained in B. If $g_1, \dots, g_k$ are real valued Borel funkctions on R^n, and g is the map $g: t = (t_1, \dots, t_n) \mapsto (g_1(t), \dots, g_k(t))$, then the map $E \mapsto z(g^{-1}(E))$ from $B(R^k)$ into L is the joint observable for the observables $g_1 \circ (x_1, \dots, x_n), \dots, g_k \circ (x_1, \dots, x_n)$.

Theorem 8 enables us to define the joint distributions of $x_1, \dots, x_n$: let m be a state on L and $x_1, \dots, x_n$ observables such that the joint observable exists for them. We may then define the probability measure $P^m_{x_1, \dots, x_n}$ on $B(R^n)$ by

$$P^m_{x_1, \dots, x_n}(E) = m(z(E)), \quad E \in B(R^n).$$

The measure $P^m_{x_1, \dots, x_n}$ is called the joint probability distribution of $x_1, \dots, x_n$ in the state m. If $E \in B(R^n)$ is of the form $E = E_1 \times \dots \times E_n$ then

$$P^m_{x_1, \dots, x_n}(E) = m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\Big).$$

If $g_1, \dots, g_k$ are real valued Borel functions on R^n, then for the observables $(y_1, \dots, y_k)$, where $y_i = g_{i^\circ}(x_1, \dots, x_n)$, $i = 1, \dots, k$, we have

$$P^m_{y_1, \dots, y_k}(F) = m(z(g^{-1}(F)) = P^m_{x_1, \dots, x_n}(g^{-1}(F)) \tag{3}$$

where g is the map $g(t) = (g_1(t), \dots, g_k(t))$ from R^n to R^k. From this we see that the rules for the calculation of the probability distributions are the standard ones of the probability theory.

Let L be a lattice logic. For any observable x, $x(E)$ denotes the event that the measured value ξ of the corresponding physical quantity lies in the Borel set E. If a quantum mechanical system is in a state m, then the expression

$$p^m_{x_1, \dots, x_n}(E_1 x \dots x E_n) = m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\Big), \tag{4}$$

$$E_i \in B(R^1), \qquad i = 1, 2, \dots, n$$

denotes the probability that in the given state the simultanoeous measurement of the observables $x_1, \dots, x_n$ gives measured values ξ_i lying in Borel sets E_i, $i = 1, \dots, n$, respectively.

If the function $p^m_{x_1, \dots, x_n}$ for given $x_1, \dots, x_n$ may be extended to a probability measure on $B(R^n)$ for some state m, we say that the observables $x_1, \dots, x_n$ have the type 1 joint distribution in the state m. This type of joint distribution (j.d.) was introduced by Gudder [6]. He has shown that for given $x_1, \dots, x_n$ the type 1 j.d. exists in all states from a full set of states M iff $x_1, \dots, x_n$ are compatible. In general, the function $p^m_{x_1, \dots, x_n}$ for given $x_1, \dots, x_n$ may be extended to a probability measure on $B(R^n)$ (i) for any state $m \in M$, (ii) only for some states, (iii) for no state. According to this characterization, we say that the observables $x_1, \ldots, x_n$ are (i) compatible, (ii) partially compatible, (iii) totally incompatible. The case (ii) is the most interesting one.

Gudder [6] found the following criterion for the existence of type 1 j.d. for two observables x, y in a state m: the type 1 j.d. exists iff

$$m\Big(\bigvee_{i=1}^{\infty} x(E_i) \wedge y(F_i))\Big) = m\Big(\bigvee_{i=1}^{\infty} x(E_i) \wedge \bigvee_{i=1}^{\infty} y(F_i)\Big) \tag{5}$$

for any $E \times F = \bigcup_{i=1}^{\infty} E_i \times F_i$, where E, $F \in B(R^1)$ and $E_i \times F_i$ are disjoint measurable rectangles, $i = 1, 2, \ldots$. It was shown in [18] that this criterion can be essentially simplified.

Theorem 9. The observables x and y have a type 1 j.d. in a state m iff

(i) $m(x(E_1 \cup E_2) \wedge y(F)) = m(x(E_1) \wedge y(F)) + m(x(E_2) \wedge y(F))$ for any E_1, E_2, $F \in B(R^1)$ such that $E_1 \cap E_2 = \emptyset$, and

(ii) $m(xE) \wedge y(F_1 \cup F_2)) = m(x(E) \wedge y(F_1)) + m(x(E) \wedge y(F_2))$ for any E, F_1, $F_2 \in B(R^1)$ such that $F_1 \cap F_2 = \emptyset$.

Proof. Necessity:

(i)
$$m(x(E_1 \cup E_2) \wedge y(F)) = p^m_{x,y}((E_1 \cup E_2) \times F) = p^m_{x,y}(E_1 \times F) + \\ + p^m_{x,y}(E_2 \times F) = m(x(E_1) \wedge y(F)) + m(x(E_2) \wedge y(F)),$$

the second equality follows from the fact that $p^m_{x,y}$ is a measure. The proof of (ii) is analogical.

To prove sufficiency, let (i) and (ii) be fulfilled. Let D be the class of all measurable rectangles $E \times F$, E, $F \in B(R^1)$ and let A be the algebra of all finite, disjoint unions of the sets of D. For all $E \in D$, $E = E^1 \times E^2$, let us set

$$\mu(E) = \mu(E^1 \times E^2) = m(x(E^1) \wedge y(E^2)).$$

It can be shown by routine arguments that the set function μ: $D \to [0, 1]$ is finitely additive. Now let $E \in A$, $E = \bigcup_{i=1}^{n} E_i$, $E_i \in D$, $i = 1, 2, \ldots, n$. Let us set

$$\mu(E) = \sum_{i=1}^{n} \mu(E_i).$$

The set function μ: $A \to [0, 1]$ is well defined and additive. By [15, Cor. 6.8, p. 30], μ is σ-additive, i.e. there is a unique extension to $B(R^1)$.

This result can be generalized for any finite set $x_1, x_2, \ldots, x_n$ of observables: the observables $x_1, \ldots, x_n$ have a type 1 j. d. in a state m iff

$$m\Big(\bigwedge_{i=1}^{n} x(E_{i_1} \cup E_{i_2}) = \sum_{j_1, \ldots, j_n=1}^{2} m\Big(\bigwedge_{i=1}^{n} x_i(E_{i_{j_i}})\Big), \tag{6}$$

for any E_{i1}, E_{i_2}, where $E_{i_1} \cap E_{i_2} = \emptyset$ for $i = 1, 2, \ldots, n$.

In [2] there is shown that (6) is equivalent to the following

$$\sum_{j_1, \ldots, j_n = 0}^{1} m\Big(\bigwedge_{i=1}^{n} x_i(E_i^{j_i})\Big) = 1, \tag{7}$$

for any $E_1, \ldots, E_n \in B(R^1)$, where we set $E^j = E$ if $j = 1$, $E^j = R^1 - E$ if $j = 0$.

If an observable x has a pure point spectrum $\sigma(x) = \{\lambda_1, \lambda_2, \ldots\}$, then $x(E) = \bigvee_{\lambda_i \in E} x\{\lambda_i\}$ for any $E \in B(R^1)$. Now if the observables $x_1, \ldots, x_n$ with pure point spectra $\sigma(x_i) = \{\lambda_1^i, \lambda_2^i, \ldots\}$, $i = 1, 2, \ldots, n$, have a joint distribution in a state m, then the measure $p^m_{x_1, \ldots, x_n}$ must be concentrated on the set $\sigma(x_1) \times \sigma(x_2) \times \ldots \times \sigma(x_n)$, i.e.

$$\sum_{i_1, \ldots, i_n = 1}^{\infty} p^m_{x_1, \ldots, x_n}\{\lambda_{i_1}^1, \lambda_{i_2}^2, \ldots, \lambda_{i_n}^n\} = m\Big(\bigvee_{j_1, \ldots, j_n = 1}^{\infty} \bigwedge_{i=1}^{n} x_i\{\lambda_{j_i}^i\}\Big) = 1$$

On the other hand, let

$$m\Big(\bigvee_{j_1, \ldots, j_n = 1}^{\infty} \bigwedge_{i=1}^{n} x_i\{\lambda_{j_i}^i\}\Big) = 1$$

As

$$\bigvee_{j_1, \ldots, j_n = 1}^{\infty} \bigwedge_{i=1}^{n} x_i\{\lambda_{j_i}^i\} \leq a(E_1, \ldots, E_n)$$

for any $E_1, \ldots, E_n \in B(R^1)$, where we put

$$a(E_1, \ldots, E_n) = \bigwedge_{j_1, \ldots, j_n = 0}^{1} \bigwedge_{i=1}^{n} x_i(E_i^{j_i}),$$

then by (7) the type 1 j.d. in the state m exists. Thus we get the following criterion: the type 1 j.d. in a state m for the observables $x_1, \ldots, x_n$ with pure point spectra exists iff $m(a_0) = 1$, where

$$a_0 = \bigvee_{j_1, \ldots, j_n = 1}^{\infty} \bigwedge_{i=1}^{n} x_i\{\lambda_{j_i}^i\}. \tag{8}$$

In the following examples, simple cases of observables are shown, which have type 1 j.d. in no states and which have type 1 j.d. only in some states.

Example 7. An observable x is called simple if its spectrum $\sigma(x) \subset \{0, 1\}$. Let x, y be simple observables such that $x\{1\} = a$, $y\{1\} = b (a, b \in L)$. It can be easily verified that the condition (8) can be rewritten in the form

$$m(a \wedge b \vee a \wedge b^\perp \vee a^\perp \wedge b \vee a^\perp \wedge b^\perp) = 1. \tag{8$'$}$$

Now let L be the logic of all closed subspaces of twoo-imensional Hilbert space. Let

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then P_x and P_y are the projection operators in the directions x and y, respectively.

As $P_x \wedge P_y = P_x^\perp \wedge P_y = P_x \wedge P_y^\perp = P_x^\perp \wedge P_y^\perp = 0$, the simple observables P_x, P_y do not have joint distribution in any state.

Example 8. Let L be the logic $L(H)$ of three-dimensional Hilbert space H. Let

$$P_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then $P_x \wedge P_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_x^\perp \wedge P_y = P_x \wedge P_y^\perp = P_x^\perp \wedge P_y^\perp = 0$.

For any $x \in H$, the map $m_x(P) = (Px, P)$, $P \in L(H)$, is a state on $L(H)$. Then by (8)′, P_x and P_y have the joint distribution in the state m_x with

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and the joint distribution does not exist in the state m_y with

$$y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The logic L is separasble if any subset of mutually orthogonal elements is at most countable. If L is separable, then to any set $\{a_\alpha : \alpha \in A\} \subset L$ there is a countable subset $\{a_i\}_{i=1}^{\infty} \subset \{a_\alpha : \alpha \in A\}$ such that $\bigwedge_{\alpha \in A} a_\alpha = \bigwedge_{i=1}^{\infty} a_i$ and $\bigvee_{\alpha \in A} a_\alpha = \bigvee_{i=1}^{\infty} a_i$.

If $x_1, \ldots, x_n$ are observables on a separable logic, then the element

$$a_0 = \bigwedge_{E_1, \ldots, E_n} a(E_1, \ldots, E_n) \tag{9}$$

exists in L and the observables $x_1, \ldots, x_n$ have a type 1 j.d. in a state m iff $m(a_0) = 1$ (see [2]).

We say that $a \in L$ is the carrier of the state m_a if $m_a(b)=0$ iff $a \perp b$. If the carrier exists, it is unique.

Let us denote by Com $(x_1, \ldots, x_n)$ the set of all states m such that the type 1 j.d. for $x_1, \ldots, x_n$ exists in m. From the properties of the carrier it follows that $m_a \in \mathrm{Com}\,(x_1, \ldots, x_n)$ iff $a \leq a(E_1, \ldots, E_n)$ for any $E_1, \ldots, E_n \in B(R^1)$.

If there is a state m_a with the carrier a for any $0 \neq a$, $a \in L$, and if L is separable or the observables $x_1, \ldots, x_n$ have pure point spectra, then we get for the element a_0 defined by (9) or (8),

$$a_0 = \bigvee \{a\colon m_a \in \mathrm{Com}\,(x_1, \ldots, x_n)\}\,. \tag{10}$$

The main result concerning the type 1 j.d. is the following theorem.

Theorem 10. Let $x_1, \ldots, x_n$ be observables on a separable logic (or let $x_1, \ldots, x_n$ be observables with pure point spectra on any logic). Let $a_0 \neq 0$, where a_0 is defined by (9) (or by (8)). Then $x_j(E)$ is compatible with a_0 for any $E \in B(R^1)$ and any $j=1, 2, \ldots, n$. The maps $E \mapsto \tilde{x}_j(E) = x_j(E) \wedge a_0$, $E \in B(R^1)$, are compatible observables on the logic $L_{[0,\,a_0]} = \{b \in L\colon b \leq a_0\}$.

For the proof see [2] and [23].

From Theorem 10 we may conclude that if $p^m_{x_1, \ldots, x_n}$ can be extended to a probability measure for some state m, then the measurements of $x_1, \ldots, x_n$ can be replaced by measurements of the observables $\tilde{x}_1, \ldots, \tilde{x}_n$ on the logic $L_{[0,\,a_0]}$. Indeed, as $m(a_0)=1$ and $x_i(E) \leftrightarrow a_0$, we have $m(x_i(E)) = m(x_i(E) \wedge a_0) = m(\tilde{x}_i(E))$ for any $E \in B(R^1)$ and any $i=1, \ldots, n$. Thus we are justified to say that the measurements of $x_1, \ldots, x_n$ in the state m can be made simultaneously. If we set

$$\begin{aligned}\tau(E_1 \times \ldots \times E_n) &= \tilde{x}_1(E_1) \wedge \tilde{x}_2(E_2) \wedge \ldots \wedge \tilde{x}_n(E_n) = \\ &= x_1(E_1) \wedge x_2(E_2) \wedge \ldots \wedge x_n(E_n) \wedge a_0,\end{aligned}$$

then, because $\tilde{x}_1, \ldots, \tilde{x}_n$ are compatible, τ can be extended to a σ-homomorphism from $B(R^n)$ into $L_{[0,\,a_0]}$. τ can be treated as a weakened form of the joint observable.

It may be interesting to ask if there are some rules for the calculation of joint distributions like to that ones for compatible observables (see (3)). In general, no functions of n non-compatible observables for $n > 1$ are defined. But in some types of logics, the sums of n observables are defined [4]. A very important example is the Hilbert space logic $L(H)$ (example 1). By spectral theorem, the (bounded) observables are in one-to-one correspondence with the (bounded) self-adjoint operators. If x and y are observables with the operators A_x and A_y, respectively, then the sum $x+y$ is defined as the observable corresponding to the operator $A_x + A_y$, if it is self-adjoint. Thus, the sum of any two bounded observables on the logic $L(H)$ is defined. If $x_1, \ldots, x_n$ are bounded observables and $f_1, \ldots, f_n, g, h$ are bounded Borel functions on R^1, then e.g. the observables

$$h(x_1 + \ldots + x_n), \quad f_1(x_1) + \ldots + f_n(x_n),$$

$$g(f_1(x_1)+\ldots+f_n(x_n)),\ \text{etc.}$$

are defined. The element a_0 for the observables $x_1, \ldots, x_n$ from (9) is now a closed subspace of H. Let us denote it by H_0. As $x_1, \ldots, x_n$ are compatible with H_0, the operators $A_{x_1}, A_{x_2}, \ldots, A_{x_n}$ are reduced by it, so that they can be treated as operators on the Hilbert space H_0. Since the sums and functions of operators reduced by H_0 are also reduced by it, and

$$x_1+\ldots+x_n/H_0=x_1/H_0+\ldots+x_n/H_0$$

$$f(x)/H_0=f(x/H_0)$$

we may conclude, that the rules for calculations of joint distributions, like that defined for compatible observables by formula (3), hold for all "allowed" functions of $x_1, \ldots, x_n$. More details on the joint distributions of observables on the Hilbert space logic can be found in [3].

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SÚHRN

O KOMPATIBILITE NA KVANTOVÝCH LOGIKÁCH

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Práca podáva prehľad o využití rôznych typov kompatibility na kvantových logikách (istých zovšeobecnených priestoroch s mierou). Zameriava sa najmä na využitie kompatibility v probléme existencie Boolových podalgebier danej logiky, obsahujúcich danú množinu a v probléme existencie združeného rozdelenia na logikách. V týchto dvoch smeroch podáva tiež, okrem prehľadu, niektoré nové výsledky.

РЕЗЮМЕ

СОВМЕСТНАЯ НАБЛЮДАЕМОСТЬ В КВАНТОВЫХ ЛОГИКАХ

Тибор Нойбрун, Силвиа Пулманнова, Братислава

Работа является обзорной статью. Главной целью является объяснить применение разных форм совместной наблюдаемости к проблеме существования Булевских алгебр содержащихся в данной логике и к проблеме существования совместного распределения наблюдаемых на логиках. В этих двух направлениях даются также некоторые новы результаты.