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QUASICONTINUOUS PROCESSES

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Introduction

The applications of the quasicontinuity in the topology and mathematical analysis are well known. (see [2] [3] [4]) It seems that the quasicontinuity and some related types of continuities may be used in the stochastic processes. Some relations were indicated in [4]. The aim of the present paper is not to give a theory of quasicontinuous processes, but to indicate some further possibilities of the applications of the quasicontinuity.

The first part of the paper extends some known results on the quasicontinuity, the second relates them to the stochastic processes.

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Let T, P be topological spaces. The function $f\colon T \to P$ is said to be quasicontinuous at $t \in T$ (see [3]) if for any neighbourhood V of $f(t)$ and any neighbourhood U of t there exists an open set $G \subset U$, $G \neq \emptyset$ such that $f(G) \subset V$. If f is quasicontinuous at any $t \in T$, it is said to be quasicontinuous.

Functions f, $g\colon T \to P$ are said to be equally distributed at $t \in T$, if for any two open sets U, V such that $f(t) \in U$, $g(t) \in V$ we have $(f^{-1}(U) \cap g^{-1}(V))^\circ \neq \emptyset$ (A° denotes the interior of A). If f, g are equally distributed at any $t \in T$, then they are said to be equally distributed.

Proposition 1.1. f, g are equally distributed if and only if $(f^{-1}(U) \cap g^{-1}(V))^\circ \neq \emptyset$ for any two open sets U, V for which $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$ holds.

Proof. Let f, g be equally distributed and let $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$ for U, V open in P. Then there is $t \in f^{-1}(U) \cap g^{-1}(V)$, hence $f(t) \in U$, $g(t) \in V$. So $(f^{-1}(U) \cap g^{-1}(V))^\circ \neq \emptyset$.

Conversely, let $(f^{-1}(U) \cap g^{-1}(V))^\circ \neq \emptyset$ whenever $f^{-1}(U) \cap g^{-1}(V) = \emptyset$ for U, V open in P. Then taking $t \in T$ and U, V open such that $f(t) \in U$, $g(t) \in V$, we have $f^{-1}(U) \cap g^{-1}(V) \supset \{t\} \neq \emptyset$. Hence $(f^{-1}(U) \cap g^{-1}(V))^{\circ} \neq \emptyset$.

Proposition 1.2. (a) Let f be continuous, g quasicontinuous Then f, g are equally distributed.

(b) Let f, g be equally distributed and moreover let f be continuous one-to-one and open. Then g is quasicontinuous.

Proof. (a) Let $U, V \subset P$ be open and $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$. Then $f^{-1}(U)$ is a nonempty open set. Let $t \in f^{-1}(U) \cap g^{-1}(V)$. The quasicontinuity of g at t implies that a nonempty open set $G \subset f^{-1}(U)$ exists such that $g(G) \subset V$. Hence $G \subset f^{-1}(U) \cap g^{-1}(V)$ So $(f^{-1}(U) \cap g^{-1}(V))^{\circ} \neq \emptyset$.

(b) Suppose g not to be quasicontinuous at $t \in T$. Then neighbourhoods V of $g(t)$ and W of t exist such that $(g^{-1}(V) \cap W)^{\circ} = \emptyset$. Put $f(W) = H$. Then H is open containing $f(t)$. Thus

$$t \in f^{-1}(H) \cap g^{-1}(V) \neq \emptyset.$$

But

$$(f^{-1}(H) \cap g^{-1}(V))^{\circ} = (W \cap g^{-1}(V))^{\circ} = \emptyset,$$

contradicting to the fact that f, g are equally distributed.

If in (b) the conditions that f is open and one-to-one are removed then Proposition 1.2. (b) is not true.

Example 1.1. Put $T = P = (-\infty, \infty)$ with the usual topology and define

$$f(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 1\rangle, \\ t-1 & \text{if } t \in \langle 1, 2), \\ 1 & \text{if } t \in \langle 2, \infty). \end{cases}$$

$$g(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0\rangle, \\ \chi_Q & \text{where } \chi_Q \text{ is the characteristic function of the set of all national numbers in } (0, 1), \\ 1 & \text{if } t \in \langle 1, \infty). \end{cases}$$

Then f, g are equally distributed, f is continuous and g is not quasicontinuous.

Remark. It is also easily seen that (a) is not true if both f and g are supposed only to be quasicontinuous. As an example one can take $T = P = (-\infty, \infty)$, $f = \chi_{(-\infty, 0)}$, $g = \chi_{<0, \infty)}$.

The cliqish functions are frequently studied as a generalization of quasicontinuous functions if the range P is a metric space.

Recall that $f: T \to P$ (T is a topological space and P a metric space with the metric ϱ) is said to be cliquish at $t \in T$, if for any $\varepsilon > 0$ and any open set U

containing t there exists an open set $G \neq \emptyset$, $G \subset U$ such that $\varrho(f(t_1), f(t_2)) < \varepsilon$ for any $t_1, t_2 \in G$. If f is cliquish at any $t \in T$, it is said to be cliquish.

Proposition 1.2 (a) may not be proved if the assumption that g is quasicontinuous is substituted by the assumption that f is cliquish.

Example 1.2 Let $T = P = (0,1)$ with the usual topology. Put $f(t) = t$ for any $t \in (0,1)$ and let g be the Riemann function

$$g(t) = \begin{cases} \frac{1}{q} & \text{if } t \text{ is rational, } t = \frac{p}{q} \\ 0 & \text{if } t \text{ is irrational.} \end{cases}$$

Then f is continuous, g is cliquish. Putting $U = V = (0,1)$ we have $f^{-1}(U) \cap g^{-1}(V) = Q$. Hence $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$, while $(f^{-1}(U) \cap g^{-1}(V))^\circ = \emptyset$.

Proposition 1.3. Let P be a Hausdorff space and f, g: $T \to P$ equally distributed and coinciding on a dense set $S \subset T$. Then $f(t) = g(t)$ for every $t \in T$.

Proof. Let $t \in T$, U, V be open sets containing $f(t)$ and $g(t)$ respectively. Then $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$, hence $G = (f^{-1}(U) \cap g^{-1}(V))^\circ \neq \emptyset$. Choose $t_1 \in G \cap S$. We have $f(t_1) = g(t_1)$ according to the assumption. Since $f(t_1) \in U$, $g(t_1) \in V$, we have $U \cap V \neq \emptyset$. But U, V are arbitrary open neighbourhoods of $f(t)$ and $g(t)$ respectively. Since P is a Hausdorff space we have $f(t) = g(t)$.

Corollary. Let P be a Hausdorff space. Let f, g: $T \to P$ and let f be continuous and g quasicontinuous. If f, g coincide on a dense set $S \subset T$, then $f(t) = g(t)$ for any $t \in T$.

A function f: $T \to R$ ($R = (-\infty, \infty)$ with the usual topology) is said to be upper (lower) quasicontinuous at $t_0 \in T$ if for any $\varepsilon > 0$ and any neighbourhood U of t_0 there exists a nonempy open set $G \subset U$ such that for any $t \in G$ $f(t) < f(t_0) + \varepsilon$ $(f(t) > f(t_0) - \varepsilon)$

Proposition 1.4. Let f: $T \to R$. Then

(a) If f is lower quasicontinuous we have

$$\sup f(S \cap G) = \sup f(G)$$

for any nonempty open set G and any dense set $S \subset T$.

(b) If f is upper quasicontinuous, then

$$\inf f(S \cap G) = \inf f(G)$$

for any nonempty open set G and any dense set $S \subset T$.

Proof. We prove (a). Evidently $\sup f(S \cap G) \leqq \sup f(G)$. Let $y \in f(G)$ and $\varepsilon > 0$. We have $y = f(t)$ where $t \in G$. Choose a nonempty open set $W \subset G$ such that $f(W) \subset (y - \varepsilon, \infty)$. In the set W, there is $t_1 \in S$. Hence $f(t_1) \in S$. Hence $f(t_1) > y - \varepsilon$, $f(t_1) \in f(S \cap G)$. So to any $y \in f(G)$ and any $\varepsilon > 0$ there exists an element $y_1 = f(t_1) \in f(S \cap G)$ such that $f(t_1) > y - \varepsilon$. Hence $\sup f(S \cap G) \geqq \sup f(G)$.

Corollary. Let $f\colon T \to R$ be a function which is both upper and lower quasicontinuous. Let S be a dense set in T. Then for any nonempty open set $G \subset T$ we have

$$\sup f(S \cap G) = \sup f(G)$$
$$\inf f(S \cap G) = \inf f(G)$$

Remark. If $f\colon T \to R$ is quasicontinuous then the assertion of the above Corollary is true.

The following example shows that the assertion of the Corollary does not imply the quasicontinuity of f.

Example 1.3. Put $T = P = (-\infty, \infty)$ and

$$f(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Then f is both upper and lower quasicontinuous, hence the assertion of Corollary is true, but f is not quasicontinuous at $t = 0$.

A set K in a topological space T is said to be quasiopen if $K \subset \overline{K^\circ}$.

We shall give a characterization of some types of quasicontinuous functions by means of quasiopen sets. For some more special cases such a characterization is known in the literature (see e.g. [1]). To describe from a common point of view various types of quasicontinuity we shall use notions of $\mathcal{S}$-quasicontinuity. (Such an attitude in another connections was used in [5]).

Let Z be *a* set and $\mathcal{S}$ a nonempty collection of subsets of Z such that for any $z \in Z$ there is $S \in \mathcal{S}$ such that $z \in S$.

A function $f\colon T \to Z$ (T is a topological space) is said to be $\mathcal{S}$-quasicontinuous at $t \in T$ if for any $S \in \mathcal{S}$ containing $f(t)$ and any open set U containing t there exists a nonempty open set $G \subset U$ such that $f(G) \subset S$. If f is $\mathcal{S}$-quasicontinuous at any $t \in T$, then it is said to be $\mathcal{S}$-quasicontinuous.

In a quite natural way we define the notion of $\mathcal{S}$-continuity. Thus f is $\mathcal{S}$-continuous at $t \in T$ if for any $S \in \mathcal{S}$ containing $f(t)$ there exists a neighbourhood U of t such that $f(U) \subset S$.

If Z is a topological space and $\mathcal{S}$ is the collection of all open subsets of Z, then $\mathcal{S}$-quasicontinuity is the usual quasicontinuity and $\mathcal{S}$-continutity is the usual continuity.

If $Z = (-\infty, \infty)$ and $\mathcal{S}$ is the ollection of all intervals $\langle a, \infty)$, $(-\infty, a\rangle$, then $\mathcal{S}$-quasicontinuity gives the upper (lower) quasicontinuity.

If $\mathcal{S}$ is the mentioned collection of subsets of Z, then $\mathcal{S}$ is said to satisfy the first countability axiom if to any $z \in Z$ there exists a decreasing sequence $\{S_n\}_{n=1}^{\infty}$ of subsets of $\mathcal{S}$ such that $z \in S_n$ $(n = 1, 2, \ldots)$ and such that for any $S \in \mathcal{S}$ for which $z \in S$ there is S_n with the property $S_n \subset S$.

Theorem 1.1. Let T be a regular (Hausdorff) topological space which satisfies the first countability axiom. Let $f\colon T \to Z$ and let the collection $\mathcal{S}$ of subsets of Z satisfy the first countability axiom at any $z \in Z$. Then f is $\mathcal{S}$-quasicontinuous at a point $t \in T$ if and only if a quasiopen set K containing t exists such that the restiction $f|K$ is $\mathcal{S}$-continuous at t.

Proof. Let K be quasiopen containing t and let $f|K$ be $\mathcal{S}$-continuous at t. Let $S \in \mathcal{S}$ be such that $f(t) \in S$ and let U be open containing t. The $\mathcal{S}$-continuity of $f|K$ at t implies that an open neighbourhood U_1 of t exists such that $U_1 \subset U$ and $f(U_1 \cap K) \subset S$. Since $U_1 \cap K \neq \emptyset$, U_1 is open and K is quasiopen, we obtain that a nonempty set $G \subset U_1 \cap K$ exists. Thus

$$G \subset U,\ f(G) \subset f(U_1 \cap K) \subset S.$$

The $\mathcal{S}$-quasicontinuity of f at t is proved

Now let f be $\mathcal{S}$-quasicontinuous at t. Let $\{S_n\}_{n=1}^{\infty}$ be the countable collection, from the first countability axiom, for the point $f(t)$. Let $\{U_n\}_{n=1}^{\infty}$ be a countable base at t with $\bar{U}_{n+1} \subset U_n$ for $n = 1, 2 \ldots$ Choosing S_1 and U_1 we obtain from the $\mathcal{S}$-quasicontinuity at t that there exists a nonempty open set $G_1 \subset U_1$ such that $f(G_1) \subset S_1$. If $G_1 = \{t\}$ then the proof of the theorem is trivial. So we may suppose $G_1 \neq \{t\}$. Put $U_{n_1} = U_1$. Then there exists an element U_{n_2} $(n_2 > n_1)$ of the basis $\{U_n\}_{n=1}^{\infty}$ such that $G_1 \cap (U_1 - \bar{U}_{n_2}) \neq \emptyset$. By the induction we can choose a subsequence $\{U_{n_i}\}_{i=1}^{\infty}$ and $G_i \neq \emptyset$ $(i = 1, 2, \ldots)$, G_i open, $G_i \subset U_{n_i}$, such that

$$f(G_i) \subset S_i \quad \text{and} \quad G_i \cap (U_{n_i} - \bar{U}_{n_{i+1}}) \neq \emptyset$$

(Note that in the case when $U_n = U_{n+k}$ for every k, the proof is obvious.)

For the simplicity write W_i instead of U_{n_i}. Further put

$$= \bigcup_{i=1}^{\infty} (G_i \cap (W_i - \bar{W}_{i+1}) \cup \{t\}.$$

It is obvious that K is quasiopen. Let $S \in \mathcal{S}$ contains $f(t)$ and let S_k be an element of the sequence $\{S_n\}_{n=1}^{\infty}$ such that $S_k \subset S$. Let $n > k$. Choose the neighbourhood W_n of t. We have

$$W_n \cap K \subset \bigcup_{i=n}^{\infty} (G_i \cap (W_i - \bar{W}_{i+1})) \cup \{t\} \subset \bigcup_{i=n}^{\infty} G_i \cup \{t\},$$

$$f(W_n \cap K) \subset \bigcup_{i=n}^{\infty} f(G_n) \cup \{f(t)\} \subset S_n \subset S_k.$$

Hence $f|K$ is $\mathcal{S}$-continuous at t.

Corollary 1. If T is a regular first countable topological space and $f\colon T \to (-\infty, \infty)$ upper (lower) quasicontinuous function at t then there exists a quasiopen set K containing t such that $f|K$ is upper (lower) continuous at t.

Note that the upper (lower) continuity is used for the simplicity to denote the notion which is sometime called upper (lower) semicontinuity.

Corollary 2. If T is a regular first countable topological space and $f: T \to (-\infty, \infty)$ a quasicontinuous function at t then there exists a quasiopen set containing t such that $f \mid K$ is continuous at t.

If f is a quasicontinuous function at a point $t \in T$ and K a quasiopen set containing t such that $f \mid K$ is continuous at t, then K is said to be a representing set of the quasicontinuity of f at the point t.

Given a collection f_α $(\alpha \in A)$ of quasicontinuous functions, we say that the collection is comparably quasicontinuous at t, if there exists a quasiopen set K containing t such that K is a representing set for the quasicontinuity at t for every f_α $(\alpha \in A)$.

2.

If nothing else is said in thlis part T denotes a topological space which is separable, first countable and such that each subspace of T is separable. Further $(\Omega, \mathcal{S}, P)$ is supposed to be a probability space, $\mathcal{N}$ the collection of all sets $N \in \mathcal{S}$ with $P(N) = 0$.

A collection $\{X_t: t \in T\}$ of random variables (real $\mathcal{S}$-measurable functions) which are defined on Ω is said to be a stochastic process. The last may be considered as a function $X(t, \omega)$, of two variables, on $T \times \Omega$.

In an analogical way to the continuous stochastic processes (see [6]) we define quasicontinuous stochastic processes.

A stochastic process is said to be quasicontinuous (continuous) if $X(t, \omega)$ is a quasicontinuous (continuous) function of the variable t for every $\omega \in \Omega$. It is said almost surely quasicontinuous (almost surely continuous) if $X(t, \omega)$ is quasicontinuous (continuous for each $\omega \notin N$, where N is some set belonging to $\mathcal{N}$.

Two processes X, Y are said to be equivalent (see [6] p. 1) if there exists for each $t \in T$ a set $N_t \in \mathcal{N}$ such that $X(t, \omega) = Y(t, \omega)$ for every $\omega \notin N_t$. They are said almost surely equal if there is a set $N \in \mathcal{S}$ such that $X(t, \omega) = Y(t, \omega)$ for every every $\omega \notin N$.

Further we say that X, Y are almost surely equally distributed if there exists $N \in \mathcal{N}$ such that $X(t, \omega)$ and $Y(t, \omega)$ are equally distributed for every $\omega \notin N$.

Theorem 2.1. Let X, Y be two equivalent stochastic processes, which are almost surely equally distributed. Then X, Y are almost surely equal.

Proof. Let $N \in \mathcal{N}$ be such that $X(t, \omega)$, $Y(t, \omega)$ are equally distributed for every $\omega \notin N$. Let $S = \{s_n: n = 1, 2, \ldots\}$ be a countable dense set in T. From the equivalence of X and Y we obtain sets $N_n \in \mathcal{N}$ $(n = 1, 2, \ldots)$ such that

$$X(s_n, \omega) = Y(s_n, \omega) \text{ for every } \omega \notin N_n$$

Putting $N^*=\bigcup_{n=1}^{\infty} N_n$, we have

$$N^*\in\mathcal{N} \text{ and } X(s_n, \omega)=Y(s_n, \omega) \text{ for every } \omega\notin N^*$$

If we denote $\tilde{N}=N^*\cup N$, then $\tilde{N}\in\mathcal{N}$ and X, Y are equally distributed for every $\omega\notin\tilde{N}$. According to Proposition 1.3. $X(t, \omega)=Y(t, \omega)$ for every $t\in T$ and $\omega\notin\tilde{N}$.

Corollary. Let X be almost surely continuous, Y almost surely quasicontinuous stochastic processes. Let X, Y be equivalent. Then they are almost surely equal.

Proof. Under the assumptions there exists $N\in\mathcal{N}$ such that X is continuous and Y quasicontinuous for every $\omega\notin N$. Moreover if $S=\{s_n: n=1, 2, \ldots\}$ is any countable and dense set in T, there exist $N_n\in\mathcal{N}$ $(n=1, 2, \ldots)$ such that

$$X(s_n, \omega)=Y(s_n, \omega) \text{ for every } \omega\notin N^*=\bigcup_{n=1}^{\infty} N_n \text{ and } n=1, 2, \ldots$$

Putting $\tilde{N}=N\cup N^*$, we have $\tilde{N}\in\mathcal{N}$ and $X(t, \omega)$ and $Y(t, \omega)$ coincide on a countable set for every $\omega\notin\tilde{N}$. According to Corollary of proposition 1.3 we have $X(t, \omega)=Y(t, \omega)$ for every $\omega\notin\tilde{N}$ and every $t\in T$.

Remark. An evident consequence of Corollary is the well known fact (see e.g. [6] p. 2) that two equivalent almost surely continuous stochastic processes are almost surely equal.

Note that the last corollary is in general not true if X, Y are supposed only to be quasicontinuous.

Example 2.1. Let $T=\Omega=\langle 0, 1\rangle$ with the Lebesgue measure and the usual topology. Put

$$X(t, \omega)=\begin{cases}0 & \text{if } t\leqq\omega,\\ 1 & \text{if } t>\omega,\end{cases}$$

$$Y(t, \omega)=\begin{cases}0 & \text{if } t<\omega,\\ 1 & \text{if } t\geqq\omega.\end{cases}$$

Then X, Y are quasicontinuous, but they are not almost surely equal.

A stochastic process X is said to be separable (compare [6] p. 26) if there exist a countable set $S\subset T$ such that for each closed interval $I\subset(-\infty, \infty)$ and every open set $G\subset T$

$$\{\omega\in\Omega: X(t, \omega)\in I \text{ for every } t\in G\} = \{\omega: X(s, \omega)\in I \text{ for every } s\in S\cap G\}$$

Sometime we say more strictly that the process is separable with restect to the countable set S.

In an obvious way an almost surely separable stochastic process X with respect to S is introduced. X is said to be almost surely separable (with respect to S) if there exists a countable dense set $S \subset T$ and a set $N \in \mathcal{N}$ such that X considered on $T \times (\Omega - N)$ is separable with respect to S.

Proposition 2.1. Let X be a quasicontinuous (almost surely quasicontinuous) stochastic process. Then X is separable (almost surely separable) with respect to any countable dense set $S \subset T$.

Proof. According to Proposition 1.4 (see also Corollary and Remark after that proposition) we have for every countable dense set $S \subset T$ and for any nonempty open set $G \subset T$

$$\inf_{t \in G} X(t, \omega) = \inf_{t \in S \cap G} X(t, \omega), \quad \sup_{t \in G} X(t, \omega) = \sup_{t \in S \cap G} X(t, \omega). \tag{1}$$

Let $I = \langle a, b \rangle$ be any closed interval. From (1) it follows that if $a \leqq X(t, \omega) \leqq b$ for every $t \in S \cap G$, then $a \leqq X(t, \omega) \leqq b$ for every $t \in G$, the converse being obvious. Thus X is separable.

Remark. From the proof we can see that it is sufficient to suppose in Proposition 2.1 the simultaneous upper and lower quasicontinuity of the process X (i.e. the upper quasicontinuity and the lower quasicontinuity of $X(t, \omega)$ for any $\omega \in \Omega$). Note further that the proof of Proposition 2.1 was given only for the quasicontinuous processes. The case of almost surely quasicontinuity is similar.

Theorem 2.2. Let X be an almost surely separable stochastic process and Y an equivalent to X almost surely continuous stochastic process. Then X and Y are almost surely equally distributed.

Proof. From the assumptions it follows that there exist a set $N \in \mathcal{N}$ and a countable set $S = \{s_n : n = 1, 2, \ldots\}$ such that the following is true:

(1) $X(s_n, \omega) = Y(s_n, \omega)$ for $n = 1, 2, \ldots, \ \omega \notin N$

(2) $Y(t, \omega)$ is continuous for $\omega \notin N$,

(3) $\{\omega \notin N : X(t, \omega) \in I$ for every $t \in G \cap S\} = \{\omega \notin N : X(t, \omega) \in I$ for every $t \in G\}$, where G is any open set.

Let $U, V \subset (-\infty, \infty)$ be open and let $\omega \notin N$. Suppose that (for ω fixed) $X^{-1}(U) \cap Y^{-1}(V) \neq \emptyset$. With no loss of generality we may suppose that $X^{-1}((a, b)) \cap Y^{-1}((c, d)) \neq \emptyset$, where $\langle a, b \rangle \subset U$ and $\langle c, d \rangle \subset V$. Let $t_0 \in X^{-1}((a, b)) \cap Y^{-1}((c, d))$. According to (3) the set $Y^{-1}((c, d))$ is open in T. Put $Y^{-1}((c, d)) = H$. From the condition (1) we have

$$X(S \cap H) = Y(S \cap H) \subset \langle c, d \rangle \subset V.$$

From the last and from (3) we obtain $X(H) \subset \langle c, d \rangle \subset V$. In particular $X(t_0, \omega) \in V$. Since $X(t_0, \omega) \in U$ too, we have $U \cap V \neq \emptyset$. Put $Z = U \cap V$. Then $H_1 = Y^{-1}(Z)$ is an open set and

$$X^{-1}(Z) \cap Y^{-1}(Z) \supset \{t_0\} \neq \emptyset.$$

By means of an analogical consideration as above, substituing both U and V by Z, we obtain a nomempty open set W such that

$$X(W) \subset Z = U \cap V, \quad Y(W) \subset Z = U \cap V.$$

Thus $(X^{-1}(U) \cap Y^{-1}(V))^{\circ} \neq \emptyset$.

Corollary. Let X be almost surely separable and Y an equivalent with X almost surely continuous stochastic process. Then X, Y are almost surely equal.

Proof. Apply Theorem 2.1.

Remark. In Corollary the condition of almost surely continuity may not be substituted by the almost surely quasicontinuity. It can bee seen from Example 2.1.

The notion of the stochastic continuity used in the theory of the stochastic processes (see [6] p. 21) may be in some cases substituted by a more general notion.

We say that a stochastic process X is stochastically quasicontinuous at t_0 if for any $\varepsilon > 0$ the function

$$\varphi_{\varepsilon}(t) = P(\{\omega: |X(t, \omega) - X(t_0, \omega)| \geqq \varepsilon\}$$

is quasicontinuous at t_0. It is said to be stochastically quasicontinuous if it is stochastically quasicontinuous at any $t \in T$.

Recall that the stochastic continuity is defined by means of the notion of continuity of the function φ_{ε}. Evidently any stochastically continuous process is stochastically quasicontinuous and the converse is not true.

The stochastic process is said to be comparably stochastically quasicontinuous at t_0 if the collection $\{\varphi_{\varepsilon}\}$ $(\varepsilon > 0)$ is comparably quasicontinuous at t_0.

It is again an obvious fact that the comparable stochastic quasicontinuity follows from the stochastic continuity and the converse is not true.

Theorem 2.3. If a stochastic proces X is almost surely separable and comparably stochastically quasicontinuous then it is almost surely separable with respect to any countable dense set $S \subset T$.

Proof. Let S be any countable dense set in T and let $t_0 \in T$. From the comparable quasicontinuity of $\{\varphi_{\varepsilon}\}$ at t_0 it follows that a quasiopen set K containing t_0 exists such that $\varphi_{\varepsilon} \mid K$ are continuous at t_0. Since K is quasiopen and T is first countable we have that a sequence $\{t_k\}_{k=1}^{\infty}$ of elements belonging to K exists such that $\lim_{k \to \infty} t_k = t_0$. Thus $\lim_{k \to \infty} \varphi_{\varepsilon}(t_k) = \varphi_{\varepsilon}(t_0)$, giving

$$\lim_{k \to \infty} P(\{\omega: |X(t_k, \omega) - X(t_0, \omega)| \geqq \varepsilon\}) = 0. \tag{1}$$

By (1) and the well known Riesz theorem there exists a subsequence of the sequence $\{t_k\}_{k=1}^{\infty}$ (we do not give a new notation for these subsequence) and a set $N_{t_0} \in \mathcal{N}$ such that

$$\lim_{k\to\infty} X(t_k, \omega) = X(t_0, \omega) \text{ if } \omega \notin N_{t_0}.$$

Since X is almost surely separable there exists a countable dense set $S_0 \subset T$ and a set $N \in \mathcal{N}$ such that for any open set G and any closed interval I

$$\{\omega \in \Omega - N: X(t, \omega) \in I \text{ for every } t \in G\} = \{\omega \in \Omega - N: X(t, \omega) \in I \text{ for every } t \in S \cap G\} \quad (2)$$

According to what was proved above, for any $s \in S_0$ there exists a set $N_s \in \mathcal{N}$ and a sequence $\{t_n^s\}_{n=1}^{\infty}$ of elements belonging to S such that

$$\lim_{n\to\infty} X(t_n^s, \omega) = X(s, \omega) \text{ if } \omega \notin N_s \quad (3)$$

Put

$$\tilde{N} = N \cup \bigcup_{n=1}^{\infty} N_s$$

We have $\tilde{N} \in \mathcal{N}$. By (3) we have $X(t, \omega) \in I$ for every $t \in S_0 \cap G$ and by (2) $X(t, \omega) \in I$ for every $t \in G$. Thus the almost sure separability of X with respect to S is proved.

Corollary. (See [6] p. 37). Let X be a stochastically continuous almost surely separable stochastic process. Then it is almost surely separable with respect to any countable dense set $S \subset T$.

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SÚHRN

KVÁZISPOJITÉ PROCESY

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V práci sa ukazuje, že možno použiť kvázispojité funkcie pri štúdiu stochastických procesov. Zavádza sa pojem kvázispojitého a stochasticky kvázispojitého procesu. Dokazuje sa niekoľko tvrdení, v ktorých spojitosť sa nahradzuje kvázispojitosťou a získavajú sa tak zovšeobecnenia niektorých známych tvrdení zo stochastických procesov.

РЕЗЮМЕ

КВАЗИНЕПРЕРЫВНЫЕ ПРОЦЕССЫ

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В работе показана возможность воспользоваться квазинепрерывными функциями при изучении стохастических процессов. Введено понятие квазинепрерывного и стохастически квазинепрерывного процессов. Доказано несколько утверждений обобщающих на основании квазинепрерывности некоторые утверждения известные в теории стохастических процессов.