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NOTES ON LATTICE — VALUED MEASURES

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The present notes are inspired by the Fremlin proof [1] of the earlier Wright's results [10]. We give a new presentation of some results concerning Carathéodory's measurability [5]. These results work in so called weakly (σ, ∞)-distributive vector lattices, introduced and studied by J. M. D Wright [1]. While in these papers the authors work with concrete representations of studied spaces, we deal following D. H. Fremlin with direct algebraic methods only.

Of course, our results are more general, too. Actually we study lattice group valued measures, even a little more general objects. The algebraic methods seem to be more economical. Simultaneously we prove an extension theorem for subadditive measures and the Choquet continuity lemma for induced outer measures.

1. An analogy with the real case

Let G be a commutative, partially ordered group satisfying the implication $x \leqq y \Rightarrow x + z \leqq y + z$. G is called monotonously σ-complete, if every increasing bounded sequence $(a_i)_i$ has the supremum $\bigvee a_i$ and hence every decreasing bounded sequence $(a_i)_i$ has the infimum $\bigwedge_{a_i}$, too. Sometimes we shall assume moreover that G is a boundedly σ-complete lattice-ordered group (l-group), i.e. that G is moreover a boundedly σ-complete lattice (every bounded sequence $(a_i)_i$ has the sumpremum $\bigvee a_i$ and the infimum $\bigwedge a_i$); such a group is commutative automatically.

By a G-valued measure we mean a mapping $\mu: \mathcal{R} \to G$ defined on a ring $\mathcal{R}$ of subsets of a set X satisfying the following conditions:

1. $\mu(E) \geqq 0$ for every $E \in \mathcal{R}$; $\mu(\emptyset) = 0$.

2. μ is σ-additive, i.e. $\mu(E) = \sum_{n=1}^{\infty} \mu(E_i)$ for every $E \in \mathcal{R}$ and every sequence $(E_i)_i$ of pairwise disjoint sets of $\mathcal{R}$ such that $E = \bigcup_{i=1}^{\infty} E_i$. $\Big($The symbol $\sum_{n=1}^{\infty} a_i$ (where $a_i \geqq 0$) means $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} a_i \Big)$.

Evidently an additive non-negative mapping $\mu: \mathcal{R} \to G$ is a measure iff μ is continuous from below and from above and it is iff μ is continuous from above in $\emptyset$. On the other hand μ is additive iff $\mu(\emptyset)=0$ and $\mu(A)+\mu(B) = \mu(A \cup B)+\mu(A \cap B)$ for every $A, B \in \mathcal{R}$.

As we know the notion of a vector lattice valued measure was first studied in [6], in the case of a group in [3]. Of course, the definitive results were formulated and proved a few years later [10]. A great part of the theory can be built in full analogy with the real case.

Assumptions. In this section G is assumed to be a monotonously σ-complete, commutative partially ordered group. μ is assumed to be a mapping $\mu: \mathcal{R} \to G$ defined on an algebra of subsets of X, continuous from below and from above, subadditive, monotone and $\mu(\emptyset)=0$ (μ will be called shortly a submeasure).

As in the real case the following lemma can be proved.

Lemma 1. If $A_n, B_n \in \mathcal{R}$, $A_n \nearrow A$, $B_n \nearrow B$ (or $A_n \searrow A$, $B_n \searrow B$ resp.) and $A \subset B$, then $\bigvee \mu(A_n) \leqq \bigvee \mu(B_n)$ (or $\bigwedge \mu(A_n) \leqq \bigwedge \mu(B_n)$ resp.).

This lemma permits to define $\mathcal{R}^+ = \{B \subset X;\ \exists A_n \in \mathcal{R},\ A_n \nearrow A\}$, $\mu^+(B) = \bigvee_n \mu(A_n)$, $B \in \mathcal{R}^+$ and analogously $\mathcal{R}^-$, μ^-. In the following lemmas we list the properties of μ^+, μ^-.

Lemma 2. The sets $\mathcal{R}^+$, $\mathcal{R}^-$ are lattices, the mappings μ^+, μ^- extend μ and they are monotone and subadditive. If $A \in \mathcal{R}^+$, $B \in \mathcal{R}^-$, then $A \setminus B \in \mathcal{R}^+$, $B \setminus A \in \mathcal{R}^-$ and

$$\mu^+(A) \leqq \mu^+(A \setminus B) + \mu^+(B),$$

$$\mu^-(B) \leqq \mu^-(B \setminus A) + \mu^+(A).$$

If moreover $A \supset B$, then $\mu^+(A) \geqq \mu^-(B)$.

Proof. The assertions are evident. Observe that the last property is a consequence of the preceding one, since then $B \setminus A = \emptyset$ and $\mu^-(\emptyset) = \mu(\emptyset) = 0$.

Lemma 3. If $A_n \nearrow A$ (or $A_n \searrow A$ resp.) $A_n \in \mathcal{R}^+$ (or $A_n \in \mathcal{R}^-$ resp.) then $A \in \mathcal{R}^+$ (or $A \in \mathcal{R}^-$ resp.) and $\mu^+(A) = \bigwedge_n \mu^+(A_n)$ (or $\mu^-(A) = \bigwedge_n \mu^-(A_n)$). The map μ^+ is σ-subadditive, i.e. $A \subset \bigcup_n A_n$ implies $\mu^+(A) \leqq \sum_{n=1}^{\infty} \mu^+(A_n)$.

Proof. The first assertion can be proved by a standard way ($A_{n,i} \nearrow A_n$ implies $\bigcup_{i=1}^{n} A_{n,i} \nearrow A$), the second one is a consequence of the continuity of μ^+ from below and the subadditivity of μ^+.

2. The key — the weak σ-distributivity

Now we cannot continue in an analogy with the real case. Of course, we could put (if G is boundedly complete, which is a reasonable assumption)

$$\mu^*(A) = \bigwedge\{\mu^+(B);\, B \in \mathcal{R}^+,\, B \supset A\} \quad A \subset X$$

similarly define μ_* and study the set L of those sets $A \subset X$ for which $\mu^*(A) = \mu_*(A)$. In the real case $A \in L$ iff to every $\varepsilon > 0$ there are $B \in \mathcal{R}^+$, $C \in \mathcal{R}^-$ such that $C \subset A \subset B$ and $\mu(B \setminus C) < \varepsilon$. But this characterization is impossible in general vector lattices. Therefore we shall give a modification of the Fremlin definition.

Definition 1. We shall say that $A \in L$ if there is a bounded sequence of sequences $(a_{ij})_{j=1}^{\infty}$ such that $a_{ij} \searrow 0$ $(j \to \infty)$ and there is $x \in G$ such that for every sequence $\varphi \in N^N$ there are $B_\varphi \in \mathcal{R}^+$, $C_\varphi \in \mathcal{R}^-$ such that $C_\varphi \subset A \subset B_\varphi$ and

$$\mu^+(B_\varphi \setminus C_\varphi) \leqq \bigvee_i a_{i\varphi(i)}, \quad \mu^-(C_\varphi) \leqq x \leqq \mu^+(B_\varphi).$$

(In the real case one can put $a_{ij} = \frac{1}{ij}$. Then e.g. for $\varphi(i) = j$, $i = 1, 2, \ldots$ we obtain $\bigvee_i a_{i\varphi(i)} = 1/j$.) Now the weak σ-distributivity enables us to identify the ε-definition with the usual supremum — infimum definition.

Definition 2. Let G be a monotonously σ-complete, commutative partially ordered group. We shall say that G is weakly σ-distributive, if whenever $(a_{ij})_{i,j}$ is a bounded family of points of G such that $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1, 2, \ldots)$, then $\bigwedge_{\varphi \in N^N} \bigvee_i a_{i\varphi(i)} = 0$.

In the real case to every $\varepsilon > 0$ there is $\varphi(i)$ such that for every $j \geqq \varphi(i)$ we have $a_{ij} < \varepsilon$, hence putting $\varphi\colon i \to \varphi(i)$ $\bigvee_i a_{i\varphi(i)} \leqq \varepsilon$ for every $\varepsilon > 0$ and therefore $\bigwedge_\varphi \bigvee_i a_{i\varphi(i)} = 0$. Also every regular K space (see [9] (i.e. Riesz space with the diagonal property see [4])) is weakly σ-distributive. So the results of [6] are special cases of the results of [10].

Proposition 1. Let G be a weakly σ-distributive, commutative partially ordered group. Let $A \in L$, $x \in G$ be the element mentioned in Definition 1. Then

$$x = \bigwedge\{\mu^+(B);\, B \in \mathcal{R}^+,\, B \supset A\} = \bigvee\{\mu^-(C);\, C \in \mathcal{R}^-,\, C \subset A\}.$$

Proof. If $C \in \mathcal{R}^-$, $C \subset A$, then $C \subset B_\varphi$, hence by Lemma 2

$$\mu^-(C) \leqq \mu^+(B_\varphi) \leqq \mu^+(B_\varphi \setminus C_\varphi) + \mu^-(C_\varphi) \leqq \bigvee_i a_{i\varphi(i)} + x,$$

$$\mu^-(C) - x \leqq \bigvee_i a_{i\varphi(i)}.$$

Since the last relation holds for every φ, the weak σ-distributivity gives $\mu^-(C) - x \leqq 0$, hence the element x is an upper bound of the set

$$\{\mu^-(C);\, C \in \mathcal{R}^-,\, C \subset A\}.$$

We have to prove that x is the least upper bound of the set. Let y be another upper bound. Then $y \geqq \mu^-(C_\varphi)$, hence

$$x - y \leqq x - \mu^-(C_\varphi) \leqq \mu^+(B_\varphi) - \mu^-(C_\varphi) \leqq \mu^+(B_\varphi \setminus C_\varphi) \leqq \bigvee_i a_{i\varphi(i)}.$$

A similar argument as before implies the relation $x - y \leqq 0$ and the second formula is proved. The first one can be proved similarly.

The element x will be denoted by $\mu^*(A)$.

3. The second step — a regularity property

In the usual ε-technique one can use the equality $\varepsilon = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i}$. In the general case we can substitute this property by the following one.

Definition 3. A commutative, monotonously σ-complete, partially ordered group G satisfies the condition (P), if to every bounded family of sequences $(a_{nij})_j$ such that $a_{nij} \searrow 0$ ($j \to \infty$, $n, i = 1, 2, \ldots$) and every $a > 0$ there is a bounded sequence $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) such that for every $\varphi \in N^N$

$$a \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)} \right) \leqq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Proposition 2. Every boundedly σ-complete l-group satisfies the condition (P).

Proof. [7], Proposition 1.

Proposition 2 for vector lattices has been proved by D. H. Fremlin [1].

The group valued measure extension theorem has been proved in [7]. Therefore we present here only the main idea. Of course, this idea is due to D. H. Fremlin.

If $A_n \nearrow A$ and $A_n \in L$, then also $A_n \setminus A_{n-1} \in L$ (which is not so difficult to prove) and (with a corresponding ordering of quantifiers)

$$\mu^*(A_n \setminus A_{n-1}) \geqq \mu^-(C_n) = \mu^+(B_n) - \mu^+(B_n \setminus C_n) \geqq \mu^+(B_n) - \bigvee_i a_{ni\varphi(i)},$$

where $B_n \in \mathcal{R}^+$, $C_n \in \mathcal{R}^-$ and $B_n \supset A_n \setminus A_{n-1} \supset C_n$. Then

$$\bigvee_n \mu^*(A_n) = \bigvee_n \sum_{i=1}^{n} \mu^*(A_i \setminus A_{i-1}) \geqq$$

$$\geqq \sum_{n=1}^{\infty} \mu^+(B_n) - \mu(X) \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{ni\varphi(i)} \right) \geqq$$

$$\geqq \mu^+(\bigcup B_n) - \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

what gives an upper approximation for $\mu^*(\bigcup A_n) = \bigvee_n \mu^*(A_n)$, the construction of a lower approximation being quite easy, too.

We did not present details here since we shall return to this subject later in connection with a submeasure extension theorem.

4. Choquet's lemma

Let us return to Sections 1 and 2. We have seen (Proposition 1) that for $A \in L$ there exists

$$\mu^*(A) = \bigwedge \{\mu^+(B);\, B \supset A,\, B \in \mathscr{R}^+\}.$$

If G is boundedly complete then $\mu^*(A)$ can be defined by the formula for every $A \subset X$. For the sake of simplicity we assume that $X \in \mathscr{R}$. The Choquet lemma says that $A_n \nearrow A \Rightarrow \mu^*(A_n) \nearrow \mu^*(A)$. Let us formulate and prove this theorem in a general lattice-valued case.

Definition 4. A boundedly σ-complete l-group G is weakly (σ, ∞)-distributive, if whenever L is an infinite set and $(b_{n,\lambda})_{n \in N, \lambda \in L}$ is a bounded family of points of G such that $\bigwedge_{\lambda \in L} b_{n,\lambda}$ exists for each n, then

$$\bigvee_{n=1}^{\infty} \bigwedge_{\lambda \in L} b_{n,\lambda} = \bigwedge_{\varphi \in L^N} \bigvee_n b_{n,\varphi(n)}.$$

Proposition 3. Let G be a boundedly σ-complete l-group. Let $a_{n,i,\lambda} \in G$ $(n, i = 1, 2, \ldots, \lambda \in L)$ be such elements that $\bigwedge_{\lambda \in L} a_{n,i,\lambda} = 0$. Then to every $a > 0$ there are $a_{i,\lambda}$ bounded $(i = 1, 2, \ldots, \lambda \in L)$ such that $\bigwedge_{\lambda \in L} a_{i,\lambda} = 0$ and

$$a \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)} \right) \leqq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

The proposition can be proved similarly as Proposition 2 in [7]. Since the proof is quite large, we present here only a proof in the case that G is a linear space. In this cases it suffices to put $b_{i,\lambda}=\sum_{k=1}^{i} 2^k a_{k,i-k,\lambda}$, $a_{i,\lambda}=a\wedge b_{i,\lambda}$. Then (putting $\lambda=\varphi(i)$, $i-k=j$)

$$2^k a_{k,i-k,\varphi(i)} \leqq b_{i,\varphi(i)}, \quad a_{k,j,\varphi(k+j)} \leqq 2^{-k} \bigvee_{i=1}^{\infty} b_{i,\varphi(i)},$$

$$a\wedge\Big(\sum_{k=1}^{\infty}\bigvee_{j=1}^{\infty} a_{k,j,\varphi(j+k)}\Big)\leqq a\wedge\Big(\sum_{k=1}^{\infty} 2^{-k}\bigvee_{i=1}^{\infty} b_{i,\varphi(i)}\Big)\leqq\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

Theorem 1. Let G be a weakly (σ,∞)-distributive, boundedly complete l-group. Let μ be a measure on an algebra $\mathscr{R}$ of subsets of a set X. Then for any $A_n\subset X$, the relation $A_n\nearrow A$ implies $\mu^*(A_n)\nearrow\mu^*(A)$.

Proof. For every $B\supset A_n$, $B\in\mathscr{R}^+$ we have

$$\mu^+(B)=\mu^*(A_n)+\mu^+(B)-\mu^*(A_n).$$

Putting $L=\mathscr{R}^+$, $a_{n,i,B}=\mu^+(B)-\mu^*(A_n)$ if $B\supset A_n$, $a_{n,i,B} = \mu^+(X)-\mu^*(A_n)$ in the opposite case, we obtain

$$\bigwedge_{B\in L} a_{n,i,B}=0$$

and to every $\varphi\in L^N$ and every n there is $B_n\in\mathscr{R}^+$, $B_n\supset A_n$ such that

$$\mu^+(B_n)\leqq\mu^*(A_n)+\bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)}.$$

Put $C_n=\bigcup_{i=1}^{n} B_i$. By the induction the following relation can be easily proved:

$$\mu^+(C_n)\leqq\mu^*(A_n)+\sum_{i=1}^{n}\bigvee_{k} a_{i,k,\varphi(k+i)}.$$

By Proposition 3 there are $a_{i,j}\searrow 0$ $(j\to\infty)$ such that $\mu(X)\wedge\Big(\sum_{n=1}^{\infty}\bigvee_{i} a_{n,i,\varphi(i+n)}\Big)\leqq \bigvee_{i} a_{i,\varphi(i)}$, hence (see Lemma 3)

$$\mu^*(A)=\mu^*\Big(\bigcup_{n=1}^{\infty} A_n\Big)\leqq\mu^+\Big(\bigcup_{n=1}^{\infty} C_n\Big)=$$

$$=\bigvee_{n}\mu^+(C_n)\leqq\bigvee_{n}\mu^*(A_n)+\mu(X)\wedge\Big(\sum_{i=1}^{\infty}\bigvee_{k} a_{i,k,\varphi(k+i)}\Big)\leqq$$

$$\leqq\bigvee_{n}\mu^*(A_n)+\bigvee_{i} a_{i,\varphi(i)}.$$

Since

$$\mu^*(A) - \bigvee_n \mu^*(A_n) \leqq \bigvee_i a_{i,\varphi(i)}$$

holds for every $\varphi \in L^N$, by the weak (σ, ∞)-distributivity we have

$$\mu^*(A) - \bigvee_n \mu^*(A_n) \leqq 0.$$

The opposite inequality is evident.

Observe that there is another point of view in the theory of lattice-valued functions. Instead of algebraic means one can work with positive functionals.

Proposition 4. Let G be a boundedly complete l-group of countable type. Let the set F of all positive, linear, order-continuous functionals separates points (i.e. $x \neq y \Rightarrow \exists f \in F,\ f(x) \neq f(y)$). Let $\mu\colon \mathcal{R} \to G$ be a measure, let $\mathcal{R}$ be an algebra. Then $A_n \nearrow A \Rightarrow \mu^*(A_n) \nearrow \mu^*(A)$.

Proof. First observe that $f \circ \mu\colon \mathcal{R} \to R$ is a measure and $(f \circ \mu)^+ = f \circ \mu^+$. If $B \in \mathcal{R}^+$, $B \supset A$, then

$$f(\mu^*(A)) \leqq f(\mu^+(B)) = (f \circ \mu)^+(B),$$

hence

$$f(\mu^*(A)) \leqq (f \circ \mu)^*(A).$$

Since G has the countable type, there is a sequence $(B_n)_n$ such that $B_n \supset A$ and $\mu^+(B_n) \searrow \mu^*(A)$. Therefore

$$f(\mu^*(A)) = \bigwedge f(\mu^+(B_n)) \geqq (f \circ \mu)^*(A).$$

Hence we have obtained the equality $f \circ \mu^* = (f \circ \mu)^*$. Now $A_n \nearrow A$ implies

$$f(\mu^*(A)) = (f \circ \mu)^*(A) = \sup_n\, (f \circ \mu)^*(A_n) =$$

$$= \sup_n f(\mu^*(A_n)) = f\Big(\bigvee_n \mu^*(A_n)\Big).$$

Since the last equality holds for every $f \in F$ and F separates points of G, we obtain $\mu^*(A) = \bigvee_n \mu^*(A_n)$.

5. Carathédory's subadditivity lemma

The classical Carathéodory method starts with a measure μ, extends it to an outer measure μ^* and then restricts μ^* to the family $\bar{\mathcal{S}}$ of all sets A intersecting regularly every set $E \subset X$:

$$\mu^*(E) = \mu^*(A \cap E) + \mu^*(E \setminus A).$$

Then $\bar{\mathscr{S}}$ is a σ-algebra and $\mu^* | \bar{\mathscr{S}}$ is a measure extending μ.

In our general case the mapping $^*\mu$ may be constructed and one can prove that for every outer measure $\bar{\mathscr{S}}$ is a σ-algebra and $\mu^* | \bar{\mathscr{S}}$ is a measure. Finally if μ^* is induced by a measure μ, then one can prove that μ^* extends μ. The only problem is to prove that the induced mapping μ^* is an outer measure, i.e. that μ^* is σ-subadditive.

Theorem 2. Let G be a boundedly complete, weakly (σ, ∞)-distributive l-group, $\mu: \mathscr{R} \to G$ be a measure, $\mathscr{R}$ an algebra. Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \sum_{n=1}^{\infty} \mu^*(A_n)$$

for every $A_n \subset X$.

Proof. Evidently μ^* is subadditive. Indeed,

$$\mu^*(E) + \mu^*(F) = \bigwedge\{\mu^+(E_1);\, E_1 \in \mathscr{R}^+,\, E_1 \supset E\} +$$
$$+ \bigwedge\{\mu^+(F_1);\, F_1 \in \mathscr{R}^+,\, F_1 \supset F\} =$$
$$= \bigwedge\{\mu^+(E_1) + \mu^+(F_1);\, E_1, F_1 \in \mathscr{R}^+,\, E_1 \supset E,\, F_1 \supset F\} \geqq$$
$$\geqq \bigwedge\{\mu^+(H);\, H \in \mathscr{R}^+,\, H \supset E \cup F\} = \mu^*(E \cup F).$$

Therefore by Theorem 1 $\left(\bigcup_{i=1}^{n} A_i \nearrow \bigcup_{i=1}^{\infty} A_i\right)$

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{n=1}^{\infty} \mu^*\left(\bigcup_{i=1}^{n} A_i\right) \leqq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

The same proof can be applied also in the case of the variant of the Choquet lemma presented in Proposition 5.

Proposition 5. Let G be a boundedly complette l-group of countable type. Let the set F of all positive, linear, order-continuous functionals separate points. Let $\mu: \mathscr{R} \to G$ be a measure on an algebra $\mathscr{R}$. Then $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \sum_{n=1}^{\infty} \mu^*(A_n)$ for all $A_n \subset X$.

The main result of this section presented in Theorem 2 was first proved by T. V. Panchapagesan and Shivappa Veerappa Palled. Of course, the proof presented in the section is purely algebraic and more general. Our theorem works in l-groups while P—SVP theorem was formulated only for vector lattices.

Of course, the Carathéodory method was generalized for groups also by P. Volauf in [8]. P. Volauf proves the Carathéodory subadditivity lemma under the following assumptions: G is a complete l-group of countable type such that for every $a_{ij} \searrow 0$ $(j \to \infty)$ and every $b \geqq 0$ from the relation $b \leqq \bigvee_i a_{i\varphi(i)}$ for every $\varphi \in N^N$ it follows that $b = 0$. Evidently G is weakly σ-distributive and since G has countable type, G is weakly (σ, ∞)-distributive, too. Hence in vector lattices the Volauf result is a consequence of the mentioned result of Panchapagesan and Palled.

There is a paper by S. K. Kundu on the Carathéodory method in l-groups [3]. Unfortunately some of this proofs are incorrect and so his main result does not hold. He supposes only (in our notations) that G is a boundedly complete l-group, $\mu: \mathcal{R} \to G$ is σ-subadditive, monotone, $\mu(0) = 0$, continuous from below and from above and defined on an algebra. (It is interesting that the assumptions are the same as our ones listed in Section 1.)

Theorem 4.1 of [3] asserts that to every $A \subset X$ there is $C \in \mathcal{R}$ such that $C \subset A$ and $\mu^+(C) = \mu^*(A)$. This assumption does not hold. We can see it in the following example.

Let $\mathcal{R}$ be the ring generated by the family $\{\langle a, b); a, b \in R, 0 \leqq a < b \leqq 1\}$. Put $A = \left(\frac{1}{4}, \frac{1}{2}\right)$. Then for $C \subset A$, $C \in \mathcal{R}$, $C = \bigcup_{i=1}^{n} \langle \alpha_i, \beta_i)$ $\langle \alpha_i, \beta_i)$ being pairwise disjoint we obtain

$$\mu(C) = \sum_{i=1}^{n} (\beta_i - \alpha_i) < \frac{1}{2} - \frac{1}{4} = \mu^*(A).$$

Hence there is no $C \in \mathcal{R}$, $C \subset A$ with $\mu(C) = \mu^*(A)$. The error is situated in the choice of C, since the set $C = \bigcup\{B; B \subset A, B \in \mathcal{R}\}$ need not belong to $\mathcal{R}$.

The second error can be found in Theorem 5.2 asserting that to every $A \in \mathcal{S}$ there is $B \in \mathcal{R}$ such that $A \subset B$ and $\mu^*(A) = \mu(B)$. Consider the preceding example and put A the set of all rational numbers in $\langle 0, 1)$. Then $\mu^*(A) = 0$ but $B \in \mathcal{R}$, $B \supset A$ implies $B = \langle 0, 1)$, hence $\mu(B) = 1$. There is an error is in the proof of Theorem 5.1. It is not true that

$$\mu^*(A) = \bigwedge\{\mu(B); A \subset B, B \in \mathcal{R}\}.$$

It holds only the inequality

$$\mu^*(A) = \inf\{\mu^+(C); A \subset C, C \in \mathcal{R}^+\} \leqq \inf\{\mu(B); A \subset B, B \in \mathcal{R}\}.$$

Finally, when the Kundu result would hold, the measure extension theorem would be true in every boundedly complete vector lattice. But we know that the weak σ-distributivity of G is not only a sufficient, but also a necessary condition for extending every G-valued measure.

6. Submeasures

Definition 5. Let $\mu: \mathcal{R} \to G$ be a submeasure (see Assumptions in Section 1). We shall say that μ is exhausting, if for every sequence $(D_n)_n$, $D_n \in \mathcal{R}$, $D_n \subset D_{n+1}$

(or $D_n \supset D_{n+1}$ resp.) it holds

$$\bigwedge_j \bigvee_i \mu(D_i \setminus D_j) = 0 \left(\text{or } \bigwedge_i \bigvee_j \mu(D_i \setminus D_j) = 0 \text{ resp.}\right).$$

Theorem 3. Let G be a monotonously σ-complete, weakly σ-distributive, commutative, partially ordered group satisfying the condition (P) (Definition 3). Let $\mathcal{R}$ be an algebra of subsets of a set X and let $\mu: \mathcal{R} \to G$ be an exhausting submeasure. Then on the σ-algebra $\sigma(\mathcal{R})$ generated by $\mathcal{R}$ there is exactly one submeasure $\bar{\mu}: \sigma(\mathcal{R}) \to G$ extending μ.

Proof. In Sections 2 and 3 we have constructed an extension $\mu^*: L \to G$. Now we prove that L is a monotone family, i.e. $A_n \nearrow A$ (or $A_n \searrow A$ resp.), $A_n \in L$ $(n = 1, 2, \ldots)$ implies $A \in L$.

Let $A_n \in L$ $(n = 1, 2, \ldots)$, $A_n \nearrow A$, $a_{n,i,j} \in G$ be such that $a_{n,i,j} \searrow 0$ $(j \to \infty)$. Choose $a_{i,j} \searrow 0$ $(j \to \infty)$ according to the condition (P). Let $\varphi \in N^N$, $B_n \in \mathcal{R}^+$, $C_n \in \mathcal{R}^-$, $C_n \subset A_n \subset B_n$ $(n = 1, 2, \ldots)$ and

$$\mu^+(B_n \setminus C_n) \leqq \bigvee_i a_{n,i,\varphi(i+n)}, \quad \mu^-(C_n) \leqq \mu^*(A_n) \leqq \mu^+(B_n).$$

Then

$$\bigcup_{i=1}^n C_i \in \mathcal{R}^-, \ \bigcup_{i=1}^n B_i \in \mathcal{R}^+, \ \bigcup_{i=1}^n C_i \subset A_n \subset \bigcup_{i=1}^n B_i,$$

$$\mu^+\left(\bigcup_{i=1}^n B_i \setminus \bigcup_{i=1}^n C_i\right) \leqq \sum_{k=1}^n \bigvee_i a_{k,i,\varphi(i+k)}.$$

Put $B = \bigcup_{n=1}^\infty B_n$. Then $B \in \mathcal{R}^+$ and there are $D_n \in \mathcal{R}$ such that $D_n \subset \bigcup_{i=1}^n B_i$ and $D_n \nearrow B$. Since μ is exhausting, we have

$$\bigwedge_j \mu^+(B \setminus D_j) = 0.$$

Therefore there is $a_{o,i,j} \searrow 0$ $(j \to \infty)$ such that for every $\varphi \in N^N$ there is n with

$$\mu^+(B \setminus D_n) \leqq \bigvee_i a_{o,i,\varphi(i)}.$$

We obtain

$$\mu^+\left(B \setminus \bigcup_{i=1}^n C_i\right) \leqq \mu(X) \wedge \left(\mu^+(B \setminus D_n) + \mu^+\left(\bigcup_{i=1}^n B_i \setminus \bigcup_{i=1}^n C_i\right)\right) \leqq$$

$$\leqq \mu(X) \wedge \left(\bigvee_i a_{o,i,\varphi(i)} + \sum_{k=1}^n a_{k,i,\varphi(k+i)}\right) \leqq \bigvee_i a_{i,\varphi(i)},$$

$$\mu^-\left(\bigcup_{i=1}^{n} C_i\right) \leqq \bigvee_n \mu^-(C_n) \leqq \bigvee_n \mu^*(A_n) \leqq \bigvee_n \mu^+(B_n) \leqq \mu^+(B),$$

$$\bigcup_{i=1}^{n} C_i \subset A = \bigcup_n A_n \subset B,$$

hence $A = \bigcup_{n=1}^{\infty} A_n \in L$. Moreover $\mu^*(A) = \bigvee_{n=1}^{\infty} \mu^*(A_n)$, hence $\mu^*(A_n) \nearrow \mu^*(A)$.

The implication $A_n \searrow A$, $A_n \in L$ $(n = 1, 2, ...)$ $\Rightarrow$ $A \in L$ can be proved analogously. Moreover, μ^* is on L continuous from above.

Since L is a monotone family over $\mathcal{R}$, L contains the σ-algebra $\sigma(\mathcal{R})$ generated by $\mathcal{R}$, hence $\bar{\mu} = \mu^* \mid \sigma(\mathcal{R})$ has all claimed properties.

The uniqueness of the extension is an easy consequence of the continuity of submeasures.

Remark. Note that usually exhausting submeasures are defined by another way $(D_n \subset D_{n+1} \Rightarrow \mu(D_{n+1} \setminus D_n) \to 0)$. Observe that in the so-called almost regular vector lattices (i.e. such lattices that the order convergence is equivalent with the regulator convergence) these two formulations are equivalent.

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SÚHRN

POZNÁMKY O MIERACH S HODNOTAMI VO ZVÄZOCH

Beloslav Riečan

Čisto algebraickými metódami sa dokazuje Carathéodoryho veta o rozšírení miery, veta o rozšírení submiery a Choquetova lema o spojitosti nahor vonkajšej miery indukovanej mierou. Oborom hodnôt miery je zväzovo usporiadaná grupa spĺňajúca niektoré podmienky distributívnosti.

РЕЗЮМЕ

ЗАМЕТКА О МЕРАХ С ЗНАЧЕНИЯМИ В СТРУКТУРАХ

Белослав Риечан

Чисто алгебраическими методами доказывается теорема Каратэодори о продолжении меры, теорема о продолжении полумеры и лемма Шокэ о непрерывности вверх внешней меры порожденной мерой. Областью значений меры является структурно упорядоченная группа выполняющая некоторые условия дистрибутивности.