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COMPARING MEASURE THEORETIC ENTROPY WITH TOPOLOGICAL ENTROPY FOR NONCOMPACT SPACES

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We show that the topological entropy introduced in [1] is the upper bound for the measure theoretic entropy of any weakly regular invariant probability measure.

Definition. Let $\mathcal{B}(X)$ be the σ-algebra generated by closed subsets of the Hausdorff topological space X. We say that the measure μ defined on $\mathcal{B}(X)$ is regular if for any $B \in \mathcal{B}(X)$ and $\varepsilon > 0$ there exist an open set U_ε and a closed set C_ε with $C_\varepsilon \subset B \subset U_\varepsilon$ and ${}_\mu(U_\varepsilon - C_\varepsilon) < \varepsilon$.

We say that μ is strongly regular if the sets C_ε and U_ε^c in the above definition can be choosen compact.

Let T be a continuous transformation of the topological space X. We consider the notion of a topological entropy $h(T)$ defined in [1] by means of open coverings containing finite subcoverings. For a metric space (X, d) we consider the entropy $h_d(T)$ introduced in [2] (cf. [7]).

We shall utilise definitions and methods introduced in [4].

Theorem. Let T be a continuous transformation of the Hausdorff space X. Let μ be an invariant regular probability measure on $\mathcal{B}(X)$. Then the measure theoretic entropy $h_\mu(T)$ satisfies relation:

$$h_\mu(T) \leq h(T). \tag{1}$$

If μ is invariant and strongly regular then

$$h_\mu(T) \leq h_d(T) \tag{2}$$

for any metric d generating topology on X.

Proof. For any partition $\alpha = \{A_1, \dots, A_n\}$ we construct β such that

$$H(\alpha \mid \beta) \leq 1$$

holds. We can take B_i closed subsets of A_i for $i = 1, \dots, n$ with

$$\mu(A_i - B_i) < \varepsilon \quad \text{and} \quad B_0 = X - \bigcup_{i=1}^{n} B_i.$$

Put

$$U_i = B_0 \cup B_i \quad \text{for} \quad i = 1, \ldots, n.$$

The sets U_i form the open covering $\gamma = \{U_1, \ldots, U_n\}$.

The topological conditional entropy (cf. [4])

$$H(\beta \mid \gamma) = \ln 2.$$

Let

$$h(T, \beta) = \lim_{n \to \infty} \ln \left\{ \operatorname{card} \left(\bigvee_{i=0}^{n-1} T^{-i}(\beta) \right) \right\} \qquad \text{(cf. [3], [4])}.$$

We have (cf. [4], [7])

$$h_\mu(T, \alpha) \leq h_\mu(T, \beta) + H(\alpha \mid \beta) \leq h(T, \beta) + 1 \leq$$
$$\leq h(T, \gamma) + H(\beta \mid \gamma) + 1 \leq h(T) + 1 + \ln 2.$$

Therefore

$$h_\mu(T) \leq h(T) + 1 + \ln 2.$$

Considering T^n instead of T we get

$$h_\mu(T) = \frac{1}{n} \cdot h_\mu(T) \leq \frac{1}{n} \cdot h(T^n) + \frac{[1 + \ln 2]}{n}$$

hence (1) holds.

We have utilised the equalities

$$h_\mu(T^n) = n \, . \, h\mu(T), \quad h(T^n) = n \, . \, h(T)$$

which are valid under the more general conditions (cf. [5]).

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SÚHRN

POROVNANIE PRAVDEPODOBNOSTNEJ A TOPOLOGICKEJ ENTROPIE NA NEKOMPAKTNÝCH PRIESTOROCH

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V článku je dokázané, že topologická entropia je hornou hranicou pre všetky hodnoty pravdepodobnostnej entropie slabo regulárnych pravdepodobnostných mier na ľubovoľnom Hausdorffovom priestore.

РЕЗЮМЕ

СРАВНЕНИЕ ВЕРОЯТНОСТНОЙ И ТОПОЛОГИЧЕСКОЙ ЭНТРОПИИ ДЛЯ НЕКОМПАКТНЫХ ПРОСТРАНСТВ

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В статье доказывается, что топологическая энтропия является верхной гранью для вероятностных энтропий слабо регулярных вероятностных мер на льюбом пространстве Гаусдорффа.