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ABOUT REGULAR MEASURES WITH VALUES IN ORDERED SPACE

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The paper contains a proof of a lemma of J. D. M. Wright ([1], Lemma 2.1). Although the lemma holds, its original proof is incorrect, as was remitted by T. V. Panchapagesan and Shivapa Veerappa Paled [2].

First some notions and notations.

A vector lattice X is boundedly complete if each subset of X which is bounded above has the least upper bound.

Let L be any set, M the set of all non-empty finite subsets of L, M^N the set of all functions $\Phi: N \to M$. If $\{b_{n,\lambda}\}$ $(n=1, 2, \ldots, \lambda \in L)$ is a family of points of X indexed by $N \times L$ and $\Phi \in M^N$, then $b_{n,\Phi(n)}$ is defined to be $\bigwedge\{b_{n,\lambda}: \lambda \in \Phi(n)\}$. A vector lattice X is called weakly (σ, ∞)-*distributive*, *if for each family* $\{b_{n,\lambda}\}$ $(n=1, 2, \ldots, \lambda \in L)$ of X such that $\bigwedge_{\lambda \in L} b_{n,\lambda}$ exists for each n it is

$$\bigvee_{n=1}^{\infty} \bigwedge_{\lambda \in L} b_{n,\lambda} = \bigwedge\left\{\bigvee_{n=1}^{\infty} b_{n,\Phi(n)}, \ \in M^N\right\}.$$

Let Ω be a non-empty set and let S be a σ-algebra of subsets of Ω. Then a finite X-valued measure is a map $m: S \to X$ with the folloving properties:

(1) $m(F) \geq 0$ for each $F \in S$;
(2) $m(\emptyset) = 0$;
(3) whenever $\{F_r\}_{r=1}^{\infty}$ is a sequence of pairwise disjoint elements of S then

$$m\left(\bigcup_{n=1}^{\infty} F_n\right) = \bigvee_{n=1}^{\infty} \sum_{k=1}^{n} m(F_k).$$

Let Z be a compact Hausdorff space, S the σ-algebra of Borel subsets of Z. Then a quasi-regular X-valued Borel measure on Z is an X-valued measure m on (Z, S) such that whenever U is an open subset of Z then

$$m(U) = \bigvee\{m(K); K \subset U, K \text{ is closed}\}.$$

An X-valued Borel measure m is regular if for each $B \in S$

$$m(B) = \bigvee \{m(K);\ K \subset B,\ K \text{ is closed}\}.$$

Let a boundedly complete vector lattice X be weakly (σ, ∞)-distributive. If $\{a_{n,i,\lambda}\}$ $(n = 1, 2, \ldots, i = 1, 2, \ldots, \lambda \in L)$ is a bounded sequence of elements of X for each n such that $\bigwedge_{\lambda \in L} a_{n,i,\lambda} = 0$, then for any $a \in X$, $a > 0$ there exists a bounded sequence $\{a_{n,\lambda}\}$ $(n = 1, 2, \ldots, \lambda \in L)$ such that

$$a \wedge \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\Phi(i+n)} \leq \bigvee_{i=1}^{\infty} a_{i,\Phi(i)}$$

for each $\Phi \in M^N$ ([3]).

Proposition. Let a boundedly complete vector lattice X be weakly (σ, ∞)-distributive. Let Z be a compact Hausdorff space and m an X-valued quasi-regular Borel measure on Z. Then m is a regular Borel measure.

Proof. For each Borel set $B \subset Z$ let $U(B)$ be the family of all open sets containing B and let $L(B)$ be the family of all closed subsets of B. Let

$$m^{+}(B) = \bigwedge \{m(U) \colon U \in U(B)\}$$

$$m^{-}(B) = \bigvee \{m(F) \colon F \in L(B)\}.$$

Then

$$m^{-}(B) \leq m(B) \leq m^{+}(B).$$

If $E = \{B \subset Z \colon B \text{ is a Borel set and } m^{+}(B) = m^{-}(B)\}$, then E contains all the open subsets of Z and E is a Boolean algebra of subsets of Z.

Now we prove the following assertion:

If $\{B_n\}_{n=1}^{\infty}$ is a monotonely increasing sequence of elements of E and $B = \bigcup_{n=1}^{\infty} B_n$, then $m^{+}(B) = m^{-}(B)$.

Evidently $m(B) = m^{-}(B)$.

Let $L = \bigcup_{n=1}^{\infty} U(B_n)$ and let

$$a_{k,i,U} = m(U) - m(B_k), \text{ if } U \in U(B_k)$$

$$a_{k,i,U} = m(Z) - m(B_k), \text{ if } U \in L - U(B_k).$$

Evidently $\bigwedge_{U \in L} a_{k,i,U} = 0$

Also

$$m(U_k) \leq m(B_k) + \bigvee_{i=1}^{\infty} a_{k,i,\Phi(i+k)}.$$

Put $C_n = \bigcup_{i=1}^{n} U_i$. By induction it can be proved that

$$m(C_n) \leq m(B_n) + \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{k,i,\Phi(i+k)}$$

for each n and each $\Phi \in M^N$.

Since $m(C_n) \leq m(Z)$ so

$$m(C_n) \leq m(Z) \wedge \left(m(B_n) + \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{k,i,\Phi(i+k)} \right) \leq$$

$$\leq m(B_n) + m(Z) \wedge \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{k,i,\Phi(i+k)} \leq$$

$$\leq m(B_n) + \bigvee_{i=1}^{\infty} a_{i,\Phi(i)}$$

where $\{a_{i,E}\}_{E \in L}$ is bounded for each i and $\bigwedge_{E \in L} a_{i,E} = 0$.

Put $C = \bigcup_{n=1}^{\infty} C_n$, then

$$m(C) \leq m(B) + \bigvee_{i=1}^{\infty} a_{i,\Phi(i)}.$$

C is open, therefore $C \in E$ then

$$m^+(C) \leq m(B + \bigwedge_{\Phi \in M^N} \bigvee_{i=1}^{\infty} a_{i,\Phi(i)}.$$

From the weak (σ, ∞)-distributivity of X we obtain

$$m^+(B) \leq m^+(C) \leq m(B).$$

Then $m^+(B) = m(B)$ and therefore $B \in E$.

So E is a σ-algebra of Borel subsets of Z which contains all the open sets and thus E coincides with the Borel sets. This proves the Proposition.

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