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LAWS OF LARGE NUMBERS IN BANACH LATTICES

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This paper is concerned with random variables which take values in a separable countably order complete Banach lattice. We find sufficient conditions for a sequence of random variables to obey the strong law of large numbers with respect to the order convergence, i.e. to ensure the order convergence of the consecutive arithmetic means to 0 on a set of probability 1. In contrast with papers dealing with the validity of the strong law of large numbers with respect to a norm, the case mentioned above has been rarely discussed in literature (see [5]). Nevertheless discussing such a case may not be without interest since in many Banach lattices (and topological lattices, in general) the order convergence implies the topological one.

The paper is divided into two parts. In the first part we present the strong laws of large numbers for random variables in a separable countably order complete Banach lattice satisfying an additional assumption. We prove that Toeplitz's and Kronecker's lemmas hold in such spaces. Then we derive Kolmogorov's strong law of large numbers with respect to the ordering. We also prove that strong law of large numbers holds even without the moment conditions on the variables in question. The second part deals with identically distributed random variables.

I.

We mention the basic definitions and notions which will be used throughout the paper.

A vector lattice X is called countably order complete if every non-empty at most countable subset of X which is bounded from above has a supremum.

A Banach lattice (Banach space) X is said to satisfy the condition G_α for some $0<\alpha\leq 1$ if there exists a map G: $X\to X'$ (the topological dual of X) such that

1) $\|G(x)\|=\|x\|^\alpha$
2) $G(x)x=\|x\|^{1+\alpha}$

3) $\|G(x)-(y)\| \leqslant A\|x-y\|$

for all x, $y \in X$ and some positive A.

Let (Ω, S, P) be a probability space, X any countably order complete Banach lattice. A strongly measurable function $f: \Omega \to X$ is called a random variable. The expectation of a random variable is defined by Bochner integral.

We say that a sequence $\{f_n\}$ of random variables obeys the strong law of large numbers (SLLN for short) if

$S_n(\omega)=\frac{1}{n}\sum_{1}^{n} f_k(\omega) \to 0$ (with respect to the ordering) on a set of probability 1.

In what follows the absolute value and norm will be denoted by $|\ |$ and $\|\ Z$ respectively.

Lemma 1. If f is a random variable in a Banach lattice, then $|f|$ is a random variable and $\|f\|$ is a (real) random variable.

Proof. The first statement follows from the continuity of lattice operations, the second statement is proved in [2].

Lemma 2. Let X be a Banach lattice. If $\{x_n\}$ is an increasing sequence in X which converges to x in norm, then $x=\sup x_n$.

Proof. See [1].

Theorem 1. Let $\{f_n\}$ be a sequence of independent random variables in X, X a separable countably order complete Banach lattice. If $g_n: R^+ \to R^+$, $g_n(x)>0$ for $x>0$ are nondecreasing and such that $\frac{x}{g_n(x)}$ are nondecreasing, then the convergence of

$$\sum_{1}^{\infty} \frac{Eg_n(\|f_n\|)}{g_n(n)}$$

implies the a.s. convergence of $\sum_{1}^{\infty}\frac{f_n}{n}$ with respect to the ordering.

Proof. We have $P(\|f_n\| \geqslant n) \leqslant E\,\frac{g_n(\|f_n\|)}{g_n(n)}$ and $\sum_{1}^{\infty} P\left(\frac{\|f_n\|}{n} \geqslant 1\right) < \infty$. Define Z_n as follows:

$$Z_n = \begin{cases} \|f_n\| & \|f_n\| < n \\ 0 & \|f_n\| \geqslant n \end{cases}$$

Because of the inequality $\frac{x^2}{n^2} \leqslant \frac{g_n(x)}{g_n(n)}$ which holds for $|x|<n$, we have $EZ_n^2 \leqslant \frac{n^2}{g_n(n)}\,Eg_n(\|f_n\|)$. From this we obtain $\sum_{1}^{\infty} E\left(\frac{Z_n}{n^2}\right)^2 < \infty$. We also have

$$|EZ_n| \leqslant \frac{n}{g_n(n)}\,Eg_n(\|f_n\|).$$

It follows, by using three — series theorem that the series $\sum_1^\infty \frac{\|f_n\|}{n}$ converges a.s. Since we have $\| |x| \| = \|x\|$ in any Banach lattice and since the consecutive sums of the series $\sum_1^\infty \frac{|f_n|}{n}$ are non decreasing, lemma 2 implies that $\sum_1^\infty \frac{|f_n|}{n}$ converges a.s. with respect to the ordering.

Toeplitz's Lemma. Let X be a separable countably order complete Banach lattice. Then $x_n \overset{o}{\to} x$ implies $\frac{1}{n}\sum_1^n x_i \overset{o}{\to} x$.

Proof. There are an increasing sequence $\{y_n\}$ and a decreasing sequence $\{z_n\}$ such that $y_n \leqslant x_n \leqslant z_n$ and $y_n \uparrow x \downarrow z_n$. Since every separable countably order complete Banach lattice has order continuous norm ([1]), we have $y_n \to x$ and $z_n \to x$ in norm. From this we obtain $\frac{1}{n}\sum y_i \to x$, and $\frac{1}{n}\sum z_i \to x$, in norm, since Toeplitz's lemma holds with respect to the norm convergence. Since $\frac{1}{n}\sum y_i$ and $\frac{1}{n}\sum z_i$ are monotone sequences such that $\frac{1}{n}\sum y_i \leqslant \frac{1}{n}\sum x_i \leqslant \frac{1}{n}\sum z_i$, the result follows from lemma 2.

Kronecker's Lemma. Let X be a separable countably order complete Banach lattice. Then the o-convergence of $\sum_1^\infty \frac{x_i}{i}$ implies that $\frac{1}{n}\sum_1^n x_i \overset{o}{\to} 0$.

Proof. See [5].

Theorem 2. (Kolmogorov's law) Let X be a separable countably order complete Banach lattice satisfying the condition G_1. If $\{f_n\}$ is a sequence of independent random variables in X, with $E\, f_n = 0$ such that $\sum \frac{E\|f_n\|^2}{n^2}$ converges, then $\{f_n\}$ obeys SLLN with respect to the ordering.

Proof. Let $\varphi(t)$ be min (t, t^2), $t \geqslant 0$. Because of the inequality $\frac{n^2 t^2}{n^2} \geqslant \varphi(t)$, we have that

$$\sum E\varphi\left(\frac{\|f_n\|}{n}\right) \leqslant \sum E \frac{\|f_n\|_2}{n^2}.$$

This together with the assumptions of the theorem implies that the series $\sum \frac{\|f_n\|}{n}$ converges (see [4], th. 1). From this we obtain the a.s. convergence of $\sum \frac{|f_n|}{n}$ with respect to the ordering. The theorem follows by applying Kronecker's lemma.

Corollary 1. SLLN holds for each sequence of independent random variables with values in 1^p or L^p (provided the latter is separable), $p \geq 2$ such that $E\, f_n = 0$ and $\sum \frac{E\,\|f_n\|^2}{n^2}$ converges.

Proof. The proof that 1^p and L^p — spaces, $p \geq 2$ satisfy the condition G can be found in [3] or [4].

Theorem 3. Let f_n be independent, symmetric random variables in a separable, countably order complete Banach lattice satisfying G_α for some $0 < \alpha \leq 1$. Let φ_n be convex $\varphi_n(t) > 0$ for $t > 0$ and such that $\frac{\varphi_n(t)}{t}$ and $\frac{t^{1+\alpha}}{\varphi_n(t)}$ do not decrease. If the series $\sum \frac{E\varphi_n\,\|f_n\|}{\varphi_n\,\|f_n\| + \varphi_n(n)}$ converges, then $\{f_n\}$ obeys SLLN with respect to the ordering.

Proof. It is sufficient to show the convergence of

$$\sum E\varphi\,\frac{\|Z_n - EZ_n\|}{n},\quad \varphi(t) = \min\,(t,\, t^{1+\alpha}),\quad t \geq 0$$

with

$$Z_n = \begin{cases} f_n & \|f_n < n \\ 0 & \|f_n\| \geq n \end{cases}$$

and then apply [4], th. 1.

It is easy to check that $\varphi(t+s) \leq K(\varphi(t) + \varphi(s))$, $t, s \geq 0$ for some constant K, depending only on α. Owing to the assumptions on φ_n we have $\varphi_n(nt)\,(\varphi_n(n))^{-1} \geq \varphi(t)$, $t \geq 0$, analysing separately the case of $t \leq 1$ and $t > 1$, respectively. Consequently, we obtain

$$E\varphi\,\frac{\|Z_n - EZ_n\|}{n} \leq E\varphi\,\frac{(\|Z_n\| + \|EZ_n\|)}{n} \leq KE\Big(\varphi\,\frac{\|Z_n\|}{n} + \varphi\,\frac{\|EZ_n\|}{n}\Big) \leq$$

$$\leq 2K\,\frac{E\varphi_n\|Z_n\|}{\varphi_n(n)}.$$

We have

$$E\,\frac{\varphi_n\|f_n\|}{\varphi_n\|f_n\| + \varphi_n(n)} =$$

$$= E\,\frac{\varphi_n\|f_n\|}{\varphi_n\|f_n\| + \varphi_n(n)}\,\chi\{\|f_n\| < n\} + E\,\frac{\varphi_n\|f_n\|}{\varphi_n\|f_n\| + \varphi_n(n)}\,\chi\{\|f_n\| \geq n\} \geq$$

$$\geq \frac{E\varphi_n\|Z_n\|}{2\varphi_n(n)} + \frac{1}{2}\,P(\|f_n\| \geq n),$$

because $\varphi_n(t)(t^{-1})$ do not decrease. This yields the convergence of $\sum \frac{E\varphi_n\|Z_n\|}{\varphi_n(n)}$.

Now [4], th. 1 implies that the series $\sum \frac{\|f_n\|}{n}$ converges a.s. The rest of the proof follows by lemma 2 and Kronecker's lemma.

II.

For identically distributed random variables the following theorems can be derived from the theorem 3.

Theorem 4. Let f_n be symmetric, independent and indentically distributed random variables in a separable, countably order complete Banach lattice satisfying the condition G_α for some $0<\alpha\leq 1$. Let φ be convex, $\varphi(t)>0$ for $t>0$ and such that $\frac{\varphi(t)}{t}$ and $\frac{t^{1+\alpha}}{\varphi(t)}$ do not decrease and let $\sum_{k=n}^{\infty}\frac{1}{\varphi(k)}=0\left(\frac{n}{\varphi(n)}\right)$. If $\sum_{n=1}^{\infty} P(\|f_1\|\geq n)$, then f_n obeys SLLN with respect to the ordering.

Proof. We have, with Z_n being defined as in th. 3

$$\sum_{n=1}^{\infty}\frac{E\varphi\|Z_n\|}{\varphi(n)}=\sum_{n=1}^{\infty}\frac{1}{\varphi(n)}\sum_{k=1}^{n}E(\varphi\|f_1\|\chi\{k-1\leq\|f_1\|<k\})\leq$$

$$\sum_{k=1}^{\infty}E\varphi(\|f_1\|\chi\{k-1\leq\|f_1\|<k\})\sum_{n=k}^{\infty}\frac{1}{\varphi(n)}\leq\text{const } X$$

$$\sum_{k=1}^{\infty}\frac{k}{\varphi(k)}E\varphi(\|f_1\|\chi\{k-1\leq\|f_1\|<k\})\leq\text{const } X$$

$$\sum_{k=1}^{\infty}kP(k-1\leq\|f_1\|<k)=\text{const}\sum_{k=0}^{\infty}P(\|f_1\|\geq k)$$

owing to the assumptions on φ. From this and because of the inequality $E\varphi\frac{\|Z_n-EZ_n\|}{n}\leq\text{const}\,E\frac{\varphi\|Z_n\|}{\varphi(n)}$, we obtain the a.s. strong convergence of the series $\sum\frac{Z_n-EZ_n}{n}$ in the ordering. The rest of the proof follows the lines of the proof of theorem 3.

If the expectations of f_n are finite, we get the following theorem (symmetry is not needed here).

Theorem 5. Let f_n be independent, identically distributed random variables in a separable, countably order complete Banach lattice satisfying the condition G_α, for some $0<\alpha\leq 1$ with $Ef_n=0$. Let φ be convex, $\varphi(t)>0$ for $t>0$ and such that $\frac{\varphi(t)}{t}$ and $\frac{t^{1+\alpha}}{\varphi(t)}$ do not decrease and $\sum_{k=n}^{\infty}\frac{1}{\varphi(k)}=0\left(\frac{n}{\varphi(n)}\right)$. Then f_n obeys SLLN with respect to the ordering.

Proof. Analogous to that of theorem 4.

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РЕЗЮМЕ

ЗАКОНЫ БОЛЬШИХ ЧИСЕЛ В РЕШЕТКАХ БАНАХА

Растислав Потоцки

В работе изучаются случайные величины с значениями в сепарабельной счетно полной банаховой решетке. В первой части доказывается усиленный закон больших чисел для случая банаховой решетки обладающей свойством G_a. Во второй части изучаются одинаково распределенные случайные величины.

SÚHRN

ZÁKONY VEĽKÝCH ČÍSEL V BANACHOVÝCH ZVÄZOCH

Rastislav Potocký

Práca sa zaoberá náhodnými premennými s hodnotami v separabilnom spočítateľne úplnom Banachovom zväze. Autor dokazuje silné zákony veľkých čísel vzhľadom na konvergenciu podľa usporiadania.