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WEAK LAWS OF LARGE NUMBERS IN CERTAIN VECTOR LATTICES

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In the last twenty years many authors have devoted their attention to laws of large numbers, i.e. results concerning the convergence of the arithmetic means to 0 in one or another sense. Most of them have been interested in laws of large numbers in the strong (norm) or the weak linear topology of the range space of random variables in question. My purpose in this paper is to discuss laws of large numbers with respect to the order convergence. In two former papers [1], [5] I studied strong laws of large numbers in Banach lattices and gave some sufficient conditions for arithmetic means of random variables with values in such a lattice to converge to 0 with respect to the order on a set of probability 1. Now I intend to focus my attention on the situation for weak laws of large numbers with respect to the order. It turns out that unlike strong laws of large numbers at least for a wide class of Banach lattices sequences of random variables which satisfy weak laws of large numbers with respect to the order and in the norm topology are the same.

The paper is divided into two parts. In section 1 I give some basic definitions and prove the weak law of large numbers for Banach lattices with an order-unit. In section 2 an extension to Frechet lattices is given.

I

Definition 1.1. By a Banach lattice I understand a normed lattice, i.e. a vector lattice with a monotonous norm, complete with respect to this norm.

An element e of a Banach lattice X is said to be an order-unit if for every $x \in X$ there is a natural number k such that $-ke \leq x \leq ke$. A norm $\| \ \|$ on X is induced by an order unit e if $\|x\| = \inf\{a > 0;\ -ae \leq x \leq ae\}$ for all $x \in X$.

Definition 1.2. Let (Ω, S, P) be a probability space, X any Banach lattice. A strongly measurable function $f\colon \Omega \to X$ is called a random variable. The expectation of a random variable is defined by Bochner integral.

There is another definition of a random variable.

Definition 1.2a. A mapping $f\colon \Omega \to X$ is a random variable if there exists a sequence of elementary random variables (in sense of [1]) $\{f_n\}$ such that $f_n(\omega)$ converges relatively uniformly to $f(\omega)$ for each ω, i.e. there is $v(\omega) \in X$ such that for every n $|f_n(\omega) - f(\omega)| \leq n^{-1} v(\omega)$.

These are equivalent definitions. Let f be a random variable in sense of the definition 1.2. Then, by [3], prop. 2.2.4. there exists a sequence $\{f_n\}$ of elementary random variables which converge uniformly to f, i.e. $\| f_n(\omega) - f(\omega) \| \| \leq n^{-1}$ for all ω. Taking a subsequence if necessary we may suppose that for each ω there is an element $v(\omega)$ in X such that $|f_n(\omega) - f(\omega)| \leq n^{-1} v(\omega)$ (for details see [2]). This means that f is a random variable in sense of the definition 1.2a.

If f_n (ω) converges relatively uniformly to $f(\omega)$ for each ω, then $f_n(\omega)$ converges to $f(\omega)$ in norm, since X is a Banach lattice. The result follows from [3], prop. 2.2.3.

Definition 1.3. I say that a sequence $\{f_n\}$ of random variables obeys the weak law of large numbers (WLLN for short) with respect to the order if there exists a positive element c of X such that $\lim\limits_{n} P\{\omega;\ |n^{-1} \sum\limits_{1}^{n} f_k(\omega)| \leqq \varepsilon c\} = 1$ for each positive ε.

Proposition 1.1. Let $\{f_n\}$ be a sequence of random variables with values in a Banach lattice X. If $\{f_n\}$ obeys WLLN with respect to the order then it obeys WLLN in the norm topology.

Proof. By the assumption there exists an element $c \in X^+$ such that $\lim P\{\omega;\ |^{-1} \sum f_k(\omega)| \leq \varepsilon c\} = 1$ for each $\varepsilon > 0$. Fix $\varepsilon > 0$ and consider $\delta > 0$ such that $\delta \|c\| \leq \varepsilon$. We have

$$P\{\omega;\ \|n^{-1} \Sigma f_k(\omega)\| \leq \varepsilon\} \geq \{\omega;\ \|n^{-1} \sum f_k(\omega)\| \leq \delta \|c\|\} \geq$$
$$P\{\omega;\ |n^{-1} \Sigma f_k(\omega)| \leq \delta c\}.$$

In Banach lattices with an order-unit we can reverse this proposition.

Theorem 1.1. Let X be a Banach lattice with a norm induced by an order-unit e. If $\{f_n\}$ is a sequence of random variables in X then the weak law of large numbers holds for $\{f_n\}$ with respect to the order if and only if it holds for $\{f_n\}$ in the norm.

Proof. Follows from the equality

$$\{\omega;\ |n^{-} \sum f_k(\omega)| \leq \varepsilon e\} = \{\omega;\ \|n^{-1} \Sigma f_k(\omega)\| \leq \varepsilon\}$$

Corollary 1.1. Let X be a Banach lattice with a norm induced by an order-unit e. Let $\{f_n\}$ be a sequence of identically distributed independent random variables in X such that $E\|f_n\| < \infty$. Then $\{f_n\}$ obeys WLLN with respect to the order.

Proof. The above conditions imply the validity of the WLLN in the norm topology (see [3], th. 5.1.1). The result follows by theorem 1.1.

Definitions 1.4. Random variables f and g are said to be weakly uncorrelated if and only if $E(Tf \,.\, Tg) = E(Tf) \,.\, E(Tg)$ for each continuous linear functional T on X.

Corollary 1.2. Let X be a Banach lattice with a norm induced by an order-unit e. If $\{f_n\}$ is a sequence of weakly uncorrelated identically distributed random variables in X such that $E\|f_n\| < \infty$ then $\{f_n\}$ obyes WLLN with respect to the order.

Proof. We have, by [3], th. 5.2.1. that a sequence of identically distributed random variables in X with the finite expectations obeys WLLN in the weak topology if and only if it obeys WLLN in the norm. The result follows from theorem 1.1.

The following theorem will show that convergence in probability with respect to the order of a Banach lattice with an order-unit is a necessary and sufficient condition for the arithmetic means of non-identically distributed random variables to converge in probability in the weak topology.

Theorem 1.2. Let X be a Banach lattice with a norm induced by an order-unit. If $\{f_n\}$ is a sequence of identically distributed random variables in X such that $E\|f_n\|^r < \infty$ for some $r > 1$ and $\{A_n\}$ is a sequence of real random variables such that

$$n^{-1} \sum_{1}^{n} (E|A_k|^{\frac{r}{r-1}})^{\frac{r-1}{r}} \leqslant G$$

for all n, where G is a positive constant and $E(A_n f_n) = E(A_1 f_1)$ for each n, then for each continuous linear functional T

$$n^{-1} \sum_{1}^{n} T(A_k f_k - E(A_1 f_1)) \to 0$$

in probability iff

$$\left| n^{-1} \sum_{1}^{n} (A_k f_k - E(A_1 f_1)) \right| \to 0$$

with respect to the order in probability.

Proof. We have, by [4], th. 2.6. that under the above conditions the sequence $\{A_n f_n\}$ satisfies WLLN in the norm topology iff it satisfies WLLN in the weak linear topology. The result follows then by theorem 1.1.

II.

Definition 2.1. By a Frechet lattice X I understand a vector lattice with a monotonous F-norm (non-necessarily homogeneous), complete with respect to this F-norm.

Proposition 2.1. Let $\{f_n\}$ be a sequence of random variables with values in a Frechet lattice. If $\{f_n\}$ obeys WLLN with respect to the order then it obeys WLLN in F-norm.

Proof. The proof goes along the lines of the proof of proposition 1.1. To complete it we make use of the following property of F-norm: for each $x \in X \|a_n x\| \to 0$ whenever $a_n \to 0$. Thus given $\varepsilon > 0$ and $c \in X^+$ we find a $\delta > 0$ such that $\|\delta c\| \leq \varepsilon$.

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РЕЗЮМЕ

СЛАБЫЕ ЗАКОНЫ БОЛЬШИХ ЧИСЕЛ В НЕКОТОРЫХ ВЕКТОРНЫХ РЕШЕТКАХ

Растислав Потоцки, Братислава

В работе доказываются слабые законы больших чисел по упорядочению в решетках Банаха и Фреше. Изучается случай пространств с порядковой единицей.

SÚHRN

SLABÉ ZÁKONY VEĽKÝCH ČÍSEL V NIEKTORÝCH VEKTOROVÝCH ZVÄZOCH

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Autor v práci dokazuje slabé zákony veľkých čísel v Banachových a Frechetových zväzoch. Podrobne rozoberá prípad priestorov s poriadkovou jednotkou.