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# A WEAK LAW OF LARGE NUMBERS FOR VECTOR LATTICE—VALUED RANDOM VARIABLES

RASTISLAV POTOCKÝ, Bratislava

I continue to investigate random variables which take their values in a vector lattice. In an earlier paper [1] *I* defined the weak law of large numbers and gave some sufficient conditions for a sequence of such random variables to satisfy it. In the present case *I* am interested only in identically distributed random variables.

In what follows *I* shall consider functions with values in an Archimedean vector lattice $E$. My terminology will follow [1] and [2].

**Definition 1.** Let $(Z, S, P)$ be a probability space. A sequence $\{f_n\}$ of functions from $Z$ to $E$ converges to a function $f$ almost uniformly if for every $\varepsilon > 0$ there exists a set $A \in S$ such that $P(A) < \varepsilon$ and $\{f_n\}$ converges relatively uniformly to $f$ uniformly on $Z - A$.

**Definition 2.** A function $f$: $Z \to E$ is called a random variable if there exists a sequence $\{f_n\}$ of countably valued random variables such that $\{f_n\}$ converges to $f$ almost uniformly.

**Definition 3.** Let $E$ be an Archimedean vector lattice. $E$ has the $\sigma$-property if for every sequence $\{x_n\}$ of the positive elements of $E$ there exists a sequence $\{a_n\}$ of positive real numbers such that the sequence $\{a_n x_n\}$ is bounded.

**Proposition 1.** Let $E$ be an Archimedean vector lattice with the $\sigma$-property. Then the vector lattice of all random variables is closed with respect to the almost uniform convergence.

**Proof.** Analogous as in [3], p. 283.

**Proposition 2.** Let $E$ be a $F$-lattice. Then each random variable is a measurable map from $Z$ to $E$.

**Proof.** Since $E$ has a monotonous $F$-norm $\| \ \|$, the inequality $|f_n(z) - f(z)| \leq a_n b$ implies $\|f_n(z) - f(z)\| \leq \|a_n b\|$. Since $\|a_n b\| \to 0$ whenever $a_n \to 0$, we have that $f_n(z) \to f(z)$ in $F$-norm for each $z \in Z$ except possibly a set of probability 0. The result follows from [4], prop. 2.1.3.

**Definition 4.** A sequence $\{f_n\}$ of random variables satisfies the weak law of large numbers, if there exists an element $a \in E^+$ such that for every $\varepsilon > 0$

$$\lim_k P\left\{z;\ \left|k^{-1}\sum_{i=1}^{k} f_i(z)\right| \leqq \varepsilon a\right\} = 1.$$

**Theorem 1.** Let $E$ be a $\sigma$-complete $F$-lattice with the $\sigma$-property. If $f_n$ are pairwise independent, identically distributed, symmetric random variables in $E$, then the condition

$$\sum_{n=1}^{\infty} P\{z;\ |f_1(z)| \leqq na\}^C < \infty \text{ for some } a \in E^+$$

is sufficient for $\{f_n\}$ to satisfy the weak law of large numbers. ($C$ stands for the set somplement.)

**Proof.** In what follows we omit a set of probability 0, if necessary. For each $n$ let $\{f_n^k\}$ be a sequence of countably valued random variables converging almost uniformly to $f_n$. Because of the inequality

$$|f_n| \leqslant |f_n - f_n^k| + |f_n^k|$$

which holds for each natural $n$ and $k$ and the assumption that $E$ has the $\sigma$-property we can regard all $f_n$ as random variables in a principal ideal of $E$ (i.e. the ideal generated by a single element, say $u$, $u \in E^+$)$I_u$, $a \leqslant u$. Since the positive cone of $E$ is closed, $I_u = \bigcup_{n=1}^{\infty} \langle -nu, nu \rangle$ is a Borel set in $E$. Hence $f_n$ are pairwise independent, identically distributed and symmetric random variables in $I_u$.

Since $E$ is $\sigma$-complete vector lattice, $I_u$ equipped with the order-unit norm (i.e. the norm induced by $u$) is a Banach space (even Banach lattice). It will be denoted by $(I_u, \|\ \|_u)$. It is well-known that in such a lattice the norm-convergence and the relatively uniform convergence are equivalent.

Let us denote by $\{y_n\}_1^{\infty}$ the set of all values which the above mentioned countably valued random variables $f_n^k$ take on. Put $y_0 = u$. Consider the countable set

$A = \left\{\sum_{i=0}^{n} a_i y_i;\ n = 0, 1, \ldots\right\}$ of all linear combinations of $y_i$ with the rational coefficients $a_i$. The set

$B = \bigcap_{r \in Q} \bigcup_{a \in A} \{x \in I_u;\ |x - a| \leqslant ru\}$ is a linear subspace of $I_u$. ($Q$ stands for the set all rational numbers.) This follows from the inequalities

$$|x + y| \leqslant |x| + |y|$$

and

$$|ax - by| \leqslant |a - b|\ |x| + |b|\ |x - y|$$

It is obvious from definition 2 that all $f_n$ take on only values in $B$. Equipped with the norm $\|\ \|_u$, $B$ becomes separable Banach space. Indeed for each $x \in B$ and each

$\varepsilon > 0$ there exists and element $a \in A$ such that $\|x - a\|_u < \varepsilon$. The completeness follows from the fact that $B$ is closed in $(I_u, \| \|_u)$.

From now on this space will be denoted by $(B, \| \|_u)$. It remains to prove that $f_n$ will maintain all the properties mentioned above. Since $B$ is separable, its Borel sets are generated by open balls. Denote these Borel sets by $W_S$. For such a ball we have

$$\{x \in B;\ \|x - x_i\|_u < \varepsilon\} = \bigcup_n \{x \in B;\ \|x - x_i\|_u \leq \varepsilon(1 - n^{-1})\} =$$
$$= \bigcup_n B \cap \{x \in I_u;\ \|x - x_i\|_u \leq \varepsilon(1 - n^{-1})\} =$$
$$= B \cap \bigcup_n \{x \in I_u;\ |x - x_i| \leq \varepsilon(1 - n^{-1})u\}.$$

Denote by $W_T$ the $\sigma$-algebra generated by subsets of $B$ open with respect to the original topology. We have proved that $W_S \subset W_T$. It means that $f_n$ are pairwise independent, identically distributed and symmetric random variables in $(B, \| \|_u)$.

Consider now the random variable $f_1$. We have

$$E\|f_1\|_u \leq 1 + \sum_{n=1}^{\infty} P\{\|f_i\|_u > n\} = 1 + \sum_{n=1}^{\infty} P\{|f_1| \leq nu\}^C < \infty$$

It follows by using [4], th. 4.1.5 that $\{f_n\}$ satisfies WLLN in $B$ and consequently in $I_u$. It means that

$$\lim_n P\left\{z \in Z;\ \left\|n^{-1}\sum_1^n f_i(z)\right\|_u \leq \varepsilon\right\} = 1 \text{ for each } \varepsilon > 0,$$

since $f_n$ are symmetric. The above equality can be rewritten as follows

$$\lim_n P\left\{z \in Z; |n^{-1}\sum_1^n f_i(z)| \leq \varepsilon u\right\} = 1$$

owing to the property of the order-norm.

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*Author's address:*

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Rastislav Potocký
Katedra teórie pravdepodobnosti a matematickej štatistiky MFF UK
Mlynská dolina
842 15 Bratislava

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