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ON POINTWISE COMPLETENESS OF NONAUTONOMOUS LINEAR DELAY DIFFERENTIAL EQUATIONS

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Consider the equation

$$x'(t) = a(t)x(t) - \tau(t)), \tag{1}$$

where $a(t)$ and $\tau(t) > 0$ are continuous functions from the set R of reals to R, and let $t_0 \in R$ be given. The equation is said to be pointwise complete provided for each point $(t_1, x_1) \in R^2$ with $t_1 > t_0$ there is a continuous initial function φ: $(-\infty, t_0] \to R$ such that the solution $x(t, t_0, \varphi) = x_\varphi(t)$ of (1) generated by φ goes through the point (t_1, x_1), i.e. if $x_\varphi(t_1) = x_1$ (cf. [1], [3], [4], [5], among others). The problem of pointwise completeness is very important e.g. in the control theory, and as is pointed out by D. Myškis (cf. [3], p. 29, or [4]), the problem is only partially solved.

It is easy to find functions $a(t)$, $\tau(t)$ such that the equation (1) is pointwise complete. A wide class of such equations is formed by certain type of oscillatory equations (cf. [2]). The following example is, however, in a certain sense extremal.

Example 1. Let C be the Cantor set in the interval [0, 1] (in general, C may be any nonempty nowhere dense perfect subset of [0, 1] of the zero Lebesgue measure). Define a function $\tau(t)$ for $t \in [0, 2]$ such that $t - \tau(t) + 1$ is the Cantor singular function for $t \in [0, 1]$ (i.e. $c(t) = t - \tau(t) + 1$ is a nondecreasing continuous function $[0, 1] \to [0, 1]$ with $c(0) = 0$, $c(1) = 1$, which is constant on each interval contiguous to C in [0, 1]), and let $t - \tau(t) = 0$ for $t \in [1, 2]$. Clearly $\tau(t) > 0$ and $t - \tau(t)$ is continuous and non-decreasing for $t \in [0, 2]$. Let $a(t)$ be a continuous function $[0, 2] \to R$ with the following properties:

(i) $a(t) = 0$ for $t \in C$;

(ii) $a(t) < 0$ for $t \in (1, 2)$ with $\int_1^2 a(s)\, ds = -1$;

(iii) if $I \subset [0, 1]$ is an open interval contiguous to C let $|a(t)| < \text{mes}\,(I)$ for $t \in I$, where mes (I) is the Lebesgue measure of I, let $\int_I a(s)\, ds = 0$, and let $a(s)$ have exactly 1 zero point in I.

Clearly $a(t)=0$ only for t in a set of the zero Lebesgue measure, hence the zero set of $a(t)$ does not contain any interval.

Now consider the equation

$$\begin{aligned} x'(t) &= a(t)x(t-\tau(t)) \quad \text{for} \quad t\in[0,2], \\ x(t) &= \varphi(t) \qquad\qquad\quad\ \text{for} \quad t\in[-1,0], \end{aligned}$$

where φ is an arbitrary initial function. We show that the solution x_φ satisfies the condition $x_\varphi(2)=0$, i.e. that the equation is not pointwise complete. Indeed,

$$S=\int_0^1 a(s)\varphi(s-\tau(s))\,\mathrm{d}s=0$$

since

$$S=\sum_{n=1}^{\infty}\int_{I(n)} a(s)\varphi(s-\tau(s))\,\mathrm{d}s+\int_C a(s)\varphi(s-\tau(s))\,\mathrm{d}s$$

where $\{I(n)\}_{n=1}^{\infty}$ is an arbitrary enumeration of the intervals contiguous to C in $[0,1]$. But $\int_{I(n)} a(s)\varphi(s-\tau(s))\,\mathrm{d}s = \text{const.} \int_{I(n)} a(s)\,\mathrm{d}s=0$, for each n, and the last term of S is 0 since mes $(C)=0$. Therefore $x(1)=x(0)+\int_0^1 a(s)\varphi(s-\tau(s))\,\mathrm{d}s$ $= x(0)$, and $x(2)=x(0)+x(0)\cdot\int_1^2 a(s)\,\mathrm{d}s=0$, q.e.d.

The following theorem gives some sufficient conditions for the pointwise completeness of the equation (1).

Theorem. Let $a(t)$ be a continuous function for $t\in[t_0,\infty)$, and let $\tau(t)$ be a positive continuous function for $t\in R$. Let A be a set of zeros of $a(t)$ in $[t_0,\infty)$. Assume that at least one of the following conditions is satisfied:

(2) The set A is nowhere dense and the function $\delta(t)=t-\tau(t)$ is strictly increasing, or

(3) the set A has the zero Lebesgue measure, $\delta(t)$ is non-decreasing and absolutely continuous, and for each $\varepsilon>0$, $(t_0, t_0+\varepsilon)$ is an interval, or

(4) A is a countable set and $\delta(t)$ is non-decreasing and for each $\varepsilon>0$, $\delta(t_0, t_0+\varepsilon)$ is an interval.

Then for each $t_1>t_0$, and each $x_1\in R$ there is a continuous initial function $\varphi\colon(-\infty, t_0]\to R$ such that the solution $x_\varphi(t)$ of (1) has the property $x_\varphi(t_1)=x_1$.

Before we proceed with the proof we introduce some terminology and notation. For non-negative integer k, let δ^k denote the k-th iterate of the function δ; in particular, δ^0 is the identity function. Note that from the assumptions on $\delta(t)$, it follows that for each $t\in R$ there exists a non-negative integer k such that $\delta^k(t)\leq t_0$. The first such integer k is called the order of the point t.

By interval is always understood a non-degenerate interval. We say that an interval I is a regular interval of the 0-th order provided $I\subset(-\infty, t_0)$, and we say that I is a regular interval of the k-th order, where k is a positive integer, if I has the following three properties:

(5) $\delta^k(I)$ is an interval;
(6) $\delta^k(I)\subset(-\infty, t_0)$;
(7) $\delta^{k-1}(I)\subset[t_0, \infty)$.

Now we prove the following two lemmas.

Lemma 1. Let the assumptions of the theorem be satisfied, and let I be a regular interval. Then the complement $I\setminus A$ of A in I contains at least two disjoint regular subintervals.

Proof. Since A is closed we have $I=(I\cap A)\cup\bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}_{n=1}^{\infty}$ are relatively open subintervals of I (not necessarily different) which are disjoint with A. Remark that an interval I_n is relatively open in I provided I_n is the intersection of an open interval with I. We have $\bigcup_{n=1}^{\infty} I_n = I\setminus A$ and

$$\delta^k(I)=\delta^k(I\cap A)\cup\bigcup_{n=1}^{\infty}\delta^k(I_n)$$

where k is the order of I. It is easy to see that $\delta^k(I\setminus A)$ cannot be an interval. Indeed, if the condition (2) from Theorem is satisfied then $\delta^k(I\cap A)$ is a nowhere dense set; if (3) or (4) is satisfied, then $\delta^k(I\cap A)$ has the zero Lebesgue measure, or is countable, respectively. Since δ^k has the intermediate value property, there is some n such that $\delta^k(I_n)$ is an interval. Denote $I_n=(c, d)$. Then there is some $\lambda\in I_n$ such that both $\delta^k(c, \lambda)$ and $\delta^k(\lambda, d)$ are intervals. Now $I^1=(c, \lambda)$, $I^2=(\lambda, d)$ are the required intervals.

Lemma 2. Let the assumptions of the theorem be satisfied, and let J be a regular interval. Assume that there is some $s\geqq t_0$ such that $J\subset(\delta(s), s)$, and let J and s have the same order. Then there is an initial function φ such that for the corresponding solution x_φ of (1) we have $x_\varphi(t)=0$ for $t\in[\delta(s), s]\setminus J$, $x_\varphi(t)\geqslant 0$ for $t\in J$, and for some $r\in J$, $x_\varphi(r)>0$.

Proof. The lemma is clearly true if J is an interval of the 0-th order. In this case we have $s=t_0$, and $x_\varphi(t)=\varphi(t)$ for $t\leqslant t_0$; it suffices to choose a suitable φ.

Now assume by the induction that the lemma is true for every regular interval of the k-th order. Let J be an interval of the $(k+1)$-th order. By Lemma 1, the set $J\setminus A$ contains a regular interval I, and this interval I contains two disjoint regular subintervals I_1, I_2. Clearly, $a(t)$ does not change the sign for $t\in I_1\cup I_2$; we may assume without loss of generality that $a(t)>0$ for such t. Denote $\delta(I_1)=I_1^*$, $\delta(I_2)=I_2^*$, and $\delta(s)=s^*$. Then I_1^*, I_2^* are regular intervals of the k-th order. By the hypothesis, there are such initial functions φ_1, φ_2, that $x_{\varphi_i}(t)=0$ for $t\in[\delta(s^*), s^*]\setminus I_i^*$, $x_{\varphi_i}(t)\geq 0$ and $x_{\varphi_i}(t)\not\equiv 0$ for $t\in I_i^*$, $i=1, 2$. Assume that $r_1<r_2$ for every $r_1\in I_1$ and $r_2\in I_2$. Denote

$$b=\int_{\delta(s)}^{s} a(\xi)x_{\varphi_1}(\delta(\xi))\,\mathrm{d}\xi\Big/\int_{\delta(s)}^{s} a(\xi)x_{\varphi_2}(\delta(\xi))\,\mathrm{d}\xi,$$

and let $\varphi(t)=\varphi_1(t)-b\,.\,\varphi_2(t)$. Then for $t\in[\delta(s), s]$ we have

$$x_\varphi(t)=\int_{\delta(s)}^{t} a(\xi)x_\varphi(\delta(\xi))\,\mathrm{d}\xi=\int_{\delta(s)}^{t} a(\xi)x_{\varphi_1}(\delta(\xi))\,\mathrm{d}\xi-b\,.\int_{\delta(s)}^{t} a(\xi)x_{\varphi_2}(\delta(\xi))\,\mathrm{d}\xi.$$

Hence $x_\varphi(t)=0$ if t lies between $\delta(s)$ and I_1, $x_\varphi(t)$ is non-decreasing for $t\in I_1$, $x_\varphi(t)=\text{const}>0$ for t lying between I_1 and I_2, $x_\varphi(t)$ is non-increasing for $t\in I_2$, and finally, $x_\varphi(t)=0$ for t lying between I_2 and s. Thus the lemma is proved.

Proof of the theorem. Denote $K=[\delta(t_1), t_1]$ and let k be the order of t_1. First we show that K contains a regular interval I of the k-th order. Put

$$u=\max\{t\in K;\ \delta^{k-1}(t)\leq t_0\}. \tag{8}$$

Such u exists since $\delta^{k-1}(t_1)>t_0\geq\delta^k(t_1) = \delta^{k-1}(\delta(t_1))$. Now put $I=(u, t_1)$. Then for each $t\in I$ we have $\delta^{k-1}(t)>t_0$ (see (8)) and $\delta^k(t)\leq\delta^k(t_1)\leq t_0$ (since t_1 has order k). Moreover, $\delta^k(I) = \delta(\delta^{k-1}(I)) = \delta((t_0, \delta^{k-1}(t_1))$ is an interval, by assumptions of the theorem. Hence $I\subset K$ is an regular interval of order k.

Let $J\subset I\setminus A$ be a regular interval (see Lemma 1). Denote $J^*=\delta(J)$, $t_1^*=\delta(t_1)$. Then $J^*\subset(\delta(t_1^*), t_1^*)$ is a regular interval, and by Lemma 2 there is an initial function φ such that $x_\varphi(t)=0$ for $t\in[\delta(t_1^*), t_1^*]\setminus J^*$, $x_\varphi(t)\geq 0$, and $x_\varphi(t)\not\equiv 0$ for $t\in J^*$. But in this case,

$$x(t_1)=\int_{t_1^*}^{t_1} a(\xi)x_\varphi(\delta(\xi))\,\mathrm{d}\xi=(\operatorname*{sign}_{t\in J} a(t))\,.\int_J |a(\xi)|x_\varphi(\delta(\xi))\,\mathrm{d}\xi\neq 0.$$

Now if we replace $\varphi(t)$ by a suitable multiple const. $\varphi(t)$ of $\varphi(t)$, we obtain $x_\varphi(t_1)=x_1$, and the theorem is proved.

Note that in Example 1 the set $A=\{t;\ a(t)=0\}$ is nowhere dense with $\operatorname{mes}(A)=0$, and that $t-\tau(t)$ is nondecreasing, continuous, but not absolutely continuous. This shows that the conditions (2) and (3) in our theorem cannot be essentially weakened. The following example shows that also the assumption that $t-\tau(t)$ is nondecreasing, cannot be omitted in (4).

Example 2. Let $a(t)=1-2t$ for $t\in[0, 1]$, $\tau(t)=1-t$ for $t\in[0, 1/2]$, $\tau(t)=3t-1$ for $t\in[1/2, 1]$. Then $\delta(t)=t-\tau(t)$ is a continuous function from $[0, 1]$ onto $[-1, 0]$. Let $\varphi(t)$ be an arbitrary initial function for $t\in[-1, 0]$. We have

$$x_\varphi(1)=\int_0^1 a(\xi)\varphi(\delta(\xi))\,\mathrm{d}\xi=$$

$$= \int_0^{1/2} (1-2t)\varphi(2t-1)\,\mathrm{d}t + \int_{1/2}^{1} (1-2t)\varphi(1-2t)\,\mathrm{d}t =$$

$$= \frac{1}{2}\int_{-1}^{0} -v\varphi(v)\,\mathrm{d}v + \frac{1}{2}\int_{-1}^{0} u\varphi(u)\mathrm{d}u = 0.$$

Hence the equation (1) in this case is not pointwise complete although $a(t)$ has exactly one zero point.

REFERENCES

[1] Driver, R. D.: Ordinary and delay differential equations. Springer, Berlin 1971.

[2] Myškis, A. D.: Linejnyje differenciaľnye uravnenija s zapazdyvajuščim argumentom. Nauka, Moskva 1972.

[3] Myškis, A. D.: O nekotorych problemach teorii differenciaľnych uravnenij s otklonjajuščimsja argumentom. Uspechi matematičeskich nauk 32 (1977), No 2 (194), 173—202.

[4] Ladas, G.: Sharp conditions for oscillations caused by delays, Applicable Analysis 9 (1979), 93—98.

[5] Yorke, J. A.: Selected topics in differential delay equations. Proc. of Japan — United States seminar on ordinary differential and functional equations. Springer, Berlin 1971, p. 16—28.

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SÚHRN

O BODOVEJ ÚPLNOSTI NEAUTONÓMNYCH LINEÁRNYCH DIFERENCIÁLNYCH ROVNÍC S ONESKORENÍM

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Pre rovnicu (1) so spojitým koeficientom a spojitým kladným oneskorením sú v práci dokázané podmienky bodovej úplnosti. Na príkladoch je ukázané, že tieto podmienky nemožno zlepšiť.

РЕЗЮМЕ

О ТОЧЕЧНОЙ ПОЛНОТЕ НЕАВТОНОМНЫХ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАЗДЫВАНИЕМ

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Для уравнения (1) с непрерывным коэффициентом и непрерывным положительным запаздыванием доказываются условия точечной полноты. На примерах показывается, что эти условия невозможно улучшить.