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ON THE STABILITY OF A MODEL FOR THE BELOUSOV—ZHABOTINSKIJ REACTION

KAROL BACHRATÝ, Bratislava

In this paper, a 3-dimensional system of differential equations, which is the Weisbuch—Salomon—Atlan model for the Belousov—Zhabotinskij reaction, is considered. In [1] V. Šeda has shown the unstable properties of solutions of this model and has provided a sufficient condition for a positive equilibrium point to be stable.

The purpose of this paper is to provide other sufficient stability conditions. It is accomplished by using the theory from papers [1], [2]. Further, a sufficient condition for a positive equilibrium point of this system to be stable with respect to a certain subset of the Euclidean space, is gained, by using La-Salle's extended stability theorem [2].

1. The model in question is

$$\begin{aligned} \dot{X} &= -5K_1X + K_5C, \\ \dot{U} &= K_1X + K_3U - K_2XU - K_4U^2, \\ \dot{C} &= 2K_3U - 4K_5C, \end{aligned} \tag{1}$$

where $K_1 - K_5$ are positive real parameters representing kinetic constants and X, U, C are concrentrations, and hence, nonnegative.

The system (1) has two equilibrium points

$$a_0 = (0, 0, 0), \quad a_1 = (X_0, U_0, C_0), \tag{2}$$

where

$$\begin{aligned} X_0 &= \frac{11K_3^2}{10(K_2K_3 + 10K_1K_4)} \\ U_0 &= \frac{11K_1K_3}{K_2K_3 + 10K_1K_4} \\ C_0 &= \frac{11K_1K_3^2}{2K_5(K_2K_3 + 10K_1K_4)} \end{aligned} \tag{3}$$

and thus

$$X_0=\frac{K_3}{10K_1}\, U_0=\frac{K_5}{5K_1}\, C_0, \quad C_0=\frac{K_3}{2K_5}\, U_0. \tag{4}$$

By Theorem 2 [1], the equilibrium point a_0 of (1) is unstable.

Now the stability of a critical point a_1 will be investigated. To this aim let us introduce new variables x, u, c, by

$$\begin{aligned} X &= X_0 + x, \\ U &= U_0 + u, \\ C &= C_0 + c, \end{aligned} \tag{5}$$

Then (1) will assume the form

$$\begin{aligned} \dot{x} &= -5K_1x + K_5c, \\ \dot{u} &= (K_1 - K_2U_0)x + (K_3 - K_2X_0 - 2K_4U_0)u - K_2xu - K_4u^2, \\ \dot{c} &= 2K_3u - 4K_5c. \end{aligned} \tag{6}$$

Obviously, a_1 is transformed into the equilibrium point (0, 0, 0) of the system (6).

Denote

$$-B=\begin{pmatrix} a, & 0, & b \\ g, & d, & 0 \\ 0, & e, & f \end{pmatrix} \tag{7}$$

$$-B=\begin{bmatrix} 5K_1, & 0, & -K_5 \\ \dfrac{10K_1(K_2K_3-K_1K_4)}{K_2K_3+10K_1K_4}, & \dfrac{K_3(K_2K_3+120K_1K_4)}{10(K_2K_3+10K_1K_4)}, & 0 \\ 0, & -2K_3, & 4K_5 \end{bmatrix}$$

and

$$f(x, u)=\begin{pmatrix} 0 \\ -K_2xu-K_4u^2 \\ 0 \end{pmatrix}.$$

Then the system (6) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ u \\ c \end{pmatrix}=B\begin{pmatrix} x \\ u \\ c \end{pmatrix}+f(x, u). \tag{8}$$

Definition 1. [2]. Let A be an $n \times n$ real matrix. Then A is said to be an M-matrix if and only if the off-diagonal elements are all nonpositive and the principial minors are all positive.

Remark 1. If A is an M-matrix, then there is a positive definite diagonal matrix W such that $WA + A'W$ is positive definite [2].

Remark 2. If the off-diagonal elements of A are all nonpositive, then the condition "A is an M-matrix", by a property of the M-matrix, is equivalent to the condition, "the leading principial minors of A are all positive", that is,

$$\det \begin{bmatrix} a_{11} & \dots & a_{1i} \\ a_{21} & \dots & a_{2i} \\ \vdots & \vdots & \vdots \\ a_{i1} & \dots & a_{ii} \end{bmatrix} > 0 \qquad \text{for } i = 1, 2, \dots, n \text{ [2].}$$

Denote

$$n = K_2K_3/K_1K_4. \tag{9}$$

Lemma 1. Let $-B$ be a matrix defined by (7). Let $0 < n \leqslant 1$. Then $-B$ is an M-matrix.

Proof. The condition $0 < n \leqslant 1$ means that $K_2K_3 \leqslant K_1K_4$. Then the offdiagonal elements of $-B$ are all nonpositive. The leading principial minors of $-B$ are

i. $5K_1 > 0$,
ii. $5K_1K_3(K_2K_3 + 120K_1K_4)/10(K_2K_3 + 10K_1K_4) > 0$,
iii. $\det(-B) = 22K_1K_3K_5 > 0$.

By Remark 2, $-B$ is an M-matrix.

Q.E.D.

Lemma 2. Let $-B$ be a matrix defined by (7). Let $1 < n < 485$. Then there is a positive definite diagonal matrix W such that $W(-B) + (-B)'W$ is positive definite.

Proof. The condition $1 < n < 485$ means that

$$K_1K_4 < K_2K_3 < 485K_1K_4. \tag{10}$$

Then

$$a > 0,\ d > 0,\ f > 0,\ g > 0 \ \text{ and } \ b < 0,\ e < 0 \ \text{ in the matrix } -B. \tag{11}$$

Define

$$0 < q^3 = \frac{e^2b^2g^2}{64a^2d^2f^2} = \frac{25(K_2K_3 - K_1K_4)^2}{16(K_2K_3 + 120K_1K_4)^2} = \frac{25(n-1)^2}{16(n+120)^2}, \quad 1 < n < 485. \tag{12}$$

Further

$$\frac{\mathrm{d}}{\mathrm{d}n} q^3 = \frac{3025(n-1)}{8(n+120)^3} > 0 \ \text{ if } \ n > 1.$$

Therefore, q^3 is an increasing function of n. Then

$$0<q^3|_{n=485}=1. \tag{13}$$

By properties (11) and (13)

$$0<\mathrm{ebg}/8adf=q^{\frac{3}{2}}<1. \tag{14}$$

Define a matrix

$$W=\begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix}$$

$$w_1=\frac{1}{q^2}\frac{g^2e^2}{16ad^2f}>0, \quad w_2=\frac{1}{q}\frac{e^2}{4df}>0, \quad w_3=1, \tag{15}$$

and a matrix $G=(g_{ij})$ $i, j=1, 2, 3$

$$G=W(-B)+(-B)'W=\begin{pmatrix} \frac{1}{q^2}\frac{g^2e^2}{8d^2f}, & \frac{1}{q}\frac{e^2g}{4df}, & \frac{1}{q^2}\frac{bg^2e^2}{16ad^2f} \\ \frac{1}{q}\frac{e^2g}{4df}, & \frac{1}{q}\frac{e^2}{2f}, & e \\ \frac{1}{q^2}\frac{bg^2e^2}{16ad^2f}, & e, & 2f \end{pmatrix}. \tag{16}$$

The leading principial minors of G are

i. $g_{11}=\dfrac{1}{q^2}\dfrac{g^2e^2}{8d^2f}>0,$

ii. $\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}=\dfrac{1}{q^2}\dfrac{g^2e^4}{16d^2f^2}\left(\dfrac{1}{q}-1\right)>0,$

iii. $\det(G)=8(q^{\frac{1}{2}}-1)^2(2q^{\frac{1}{2}}+1)a^2fe^2/b^2>0.$

The matrix G is a symmetric matrix and leading principial minors of G are all positive, then G is a positiye definite matrix [4]. The matrix W defined by (15) is a positive definite diagonal matrix.

Q.E.D.

Theorem 1. If $K_2K_3<485K_1K_4$ then the equilibrium point a_1 of the system (1) is exponentially asymptotically stable.

Proof. If $K_2K_3<485K_1K_4$ then $0<n<485$. Then there is a positive definite diagonal matrix W such that $W(-B)+(-B)'W$ is positive definite, by Lemma 1 and by Remark 1 if $0<n\leq 1$, and by Lemma 2 if $1<n<485$. Then by Theorem 2 [2] the real parts of the eigenvalues of $-B$ are all positive. Therefore, the real parts of the eigenvalues of B are all negative. Then the quilibrium point (0, 0, 0) of the system (6) is exponentially asymptotically stable [3]. Therefore, the equilibrium point a_1 of the system (1) is exponentially asymptotically stable.

Q.E.D.

2. Let P be a subset of R^3 and $a_1 \in P$. Let every solution of (1) remain in P for all $t \geq t_0$, if the initial value belongs to P.

Definition 2 [2]. A positive equilibrium point a_1 of the system (1) is said to be asymptotically stable in the large with respect to the set P, if and only if

i. the equilibrium point a is stable with respect to P, namely, if for every $\varepsilon > 0$ there exists $\delta(\varepsilon; t_0)$ such that if $\|(X(t_0), U(t_0), C(t_0)) - a_1\| < \delta$ and the solution $(X(t), U(t), C(t))$ is in P, then $\|(X(t), U(t), C(t)) - a_1\| < \varepsilon$ for $t \geq t_0$,

ii. and every solution converges to a_1 as $t \to \infty$, if $(X(t_0), U(t_0), C(t_0)) \in P$.

Further, we shall assume that $1 < n < 485$.

Now define a continuously differentiable function $V(x, u, c)$

$$V(x, u, c) = (x, u, c) W \begin{pmatrix} x \\ u \\ c \end{pmatrix}, \tag{17}$$

where W is a matrix defined by (15), then

$$V(x, u, c) \geq 0 \text{ in } R^3 \text{ and } V(x, u, c) = 0 \text{ holds only for } (0, 0, 0). \tag{18}$$

The time derivative of $V(x(t), u(t), c(t))$, along the solution of (6) is

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t), u(t), c(t))|_{(6)} =$$

$$= (x, u, c)(WB + B'W) \begin{pmatrix} x \\ u \\ c \end{pmatrix} + 2(x, u, c) W f(x, u) = \tag{19}$$

$$= -(x, u, c) G \begin{pmatrix} x \\ u \\ c \end{pmatrix} - 2 w_2 u^2 (K_2 x + K_4 u),$$

where G is a matrix defined by (16). By Lemma 2 G is positive definite.

Define

$$F(x, u, c) = (x, u, c) G \begin{pmatrix} x \\ u \\ c \end{pmatrix} + 2 w_2 u^2 (K_2 x + K_4 u). \tag{20}$$

The term $2 w_2 u^2 (K_2 x + K_4 u)$ is not dependent on c. Thus $F(x, u, c)$ is a quadratic function of c. Since $g_{33} > 0$, then there is $\min_{c \in R} F(x, u, c)$. Further, $\frac{\partial}{\partial c} F(x, u, c) = 2 g_{33} c + 2 g_{23} u + 2 g_{13} x = 0$ holds only for $c = \bar{c} = \frac{-g_{23} u - g_{13} x}{g_{33}}$. Since $\frac{\partial^2}{\partial c^2} F(x, u, c) = 2 g_{33} > 0$, then

$$\min_{c\in R} F(x, u, c) = F(x, u, c)|_{c=\bar{c}} =$$

$$= d_{11}x^2 + d_{22}u^2 + 2d_{12}xu + 2w_2u^2(K_2x + K_4u), \tag{21}$$

where

$$d_{11} = g_{11} - \frac{g_{31}^2}{g_{33}}, \quad d_{22} = g_{22} - \frac{g_{32}^2}{g_{33}}, \quad d_{12} = g_{12} - \frac{g_{13}g_{23}}{g_{33}}. \tag{22}$$

Define a subset $M(\eta)$ of R^3,

$$M(\eta) = \left\{(x, u, c) : x > -\frac{K_4}{K_2}u - \eta\right\}. \tag{23}$$

Then

$$2w_2u^2(K_2x + K_4u) \geqslant -2w_2K_2\eta u^2 \text{ in } M(\eta). \tag{24}$$

Therefore

$$F(x, u, c) \geqslant \min_{c\in R} F(x, u, c) \geqslant (x, u)D\binom{x}{u} \text{ in } M(\eta), \tag{25}$$

where D is a matrix

$$D = \begin{pmatrix} d_{11}, & d_{12} \\ d_{12}, & d_{22} - 2w_2K_2\eta \end{pmatrix}. \tag{26}$$

The matrix $G = (g_{ij})$ defined by (16) is positive definite, then $d_{11} > 0$. Therefore, D is positive definite, if and only if $\det(D) > 0$, this means

$$\eta < \frac{1}{2w_2K_2}\left(d_{22} - \frac{d_{12}^2}{d_{11}}\right), \tag{27}$$

that is,

$$\eta < \frac{K_3(n+120)Q(n)}{10K_2(n+10)}, \tag{28}$$

where n is defined by (9) and

$$Q(n) = 1 - \frac{2q}{q^{\frac{1}{2}}+1} = 1 - \frac{2\sqrt[3]{\frac{25(n-1)^2}{16(n+120)^2}}}{\sqrt[3]{\frac{5(n-1)}{4(n+120)}}+1}, \quad 1 < n < 485. \tag{29}$$

Since $F(0, 0, 0) = 0$, by (25) we obtain

Lemma 3. If (28) holds, then $F(x, u, c)$ is positive definite in $M(\eta)$.

Denote

$$k = K_1/K_2, \quad L = K_3/K_4, \tag{30}$$

then

$$k < L < 485k \quad \text{and} \quad K_1K_4 < K_2K_3 < 485K_1K_4. \tag{31}$$

Therefore

$$k = K_1/K_2 < U_0 = 11K_1K_3/(K_2K_3 + 10K_1K_4) < K_3/K_4 = L \tag{32}$$

Define a set

$$H(h) = \{(X, U, C): X_1 \leqslant X \leqslant X_2,\ k \leqslant U \leqslant h,\ C_1 \leqslant C \leqslant C_2\}, \tag{33}$$

where

$$L < h,\ 0 < C_1 < (K_3/2K_5)k,\ (K_3/2K_5)h < C_2,$$

$$0 < X_1 < (K_5/5K_1)C_1,\ (K_5/5K_1)C_2 < X_2.$$

Then by Lemma 3 [1] $a_1 \in H(h)$ and every solution of (1) remain in $H(h)$ for all $t \geqslant t_0$, if the initial value belongs to $H(h)$.

Lemma 4. Let $P(h)$ be a set

$$P(h) = \left\{(X, U, C): \frac{K_3}{10K_1}\, k \leqslant X \leqslant \frac{K_3}{10K_1}\, h,\ k \leqslant U \leqslant h,\ \frac{K_3}{2K_5}\, k \leqslant C \leqslant \frac{K_3}{2K_5}\, h\right\}$$

for $h > L$. Then $a_1 \in P(h)$ and every solution of (1) remains in $P(h)$ for all $t \geqslant t_0$, if the initial value belongs to $P(h)$.

Proof. By (32) $k < U_0 < L < h$, then $(K_3/10K_1)k < X_0 < (K_3/10K_1)h$ and $(K_3/2K_5)k < C_0 < (K_3/2K_5)h$ for $h > L$. Therefore $a_1 \in P(h)$.

Let $(X(t), U(t), C(t))$, be a solution of (1). Let the initial value for $t = t_0$ belong to $P(h)$ and $t_1 > t_0$ such that this solution does not belong to $P(h)$ at t_1. Assume, that $X(t_1) < (K_3/10K_1)k$. Then there are numbers X_1, C_1 such that $X(t_1) < X_1 < (K_3/10K_1)k$ and $X_1 < (K_5/5K_1)C_1 < (K_3/10K_1)k$, $C_1 < (K_3/2K_5)k$.

Therefore $(X(t_1), U(t_1), C(t_1)) \notin H(h)$.

This can be proved similarly from the assumption $X(t_1) > (K_3/10K_1)h$, or $U(t_1) \notin [k, h]$ or $C(t_1) \notin [(K_3/2K_5)k, (K_3/2K_5)h]$. Then by Lemma 3 [1] this solution remains in $P(h)$ for all $t \geqslant t_0$, since $P(h) \subset H(h)$.

Q.E.D.

By the transformation (5) $P(h)$ is

$$P_1(h) = \{(x, u, c): (K_3/10K_1)(k - U_0) \leqslant x \leqslant (K_3/10K_1)(h - U_0),$$

$$(k - U_0) \leqslant u \leqslant (h - U_0),\ (K_3/2K_5)(k - U_0) \leqslant c \leqslant (K_3/2K_5)(h - U_0)\}. \tag{34}$$

If

$$(K_3/10K_1)(k-U_0) > -(K_4/K_2)(k-U_0) - \eta, \tag{35}$$

then $x > -(K_4/K_2)u - \eta$ in $P_1(h)$, that is $P_1(h)$ is a subset of $M(\eta)$. The condition (35) is equivalent to the condition

$$\eta > (K_1K_4/K_2^2)(n-1). \tag{36}$$

Lemma 5. $F(x, u, c)$ defined by (20) is positive definite in $P_1(h)$, if

$$\frac{10(n+10)(n-1}{n(n+120)} < Q(n), \quad 1 < n < 485, \tag{37}$$

where $Q(n)$ is defined by (29).

Proof. The condition (37) is equivalent to

$$\frac{K_1K_4(n-1)}{K_2^2} < \frac{K_3(n+120)Q(n)}{10K_2(n+10)}.$$

Then there is an $\eta > 0$ such that

$$\frac{K_1K_4(n-1)}{K_2^2} < \eta < \frac{K_3(n+120)Q(n)}{10K_2(n+10)}.$$

Therefore (36) and (28) hold then by Lemma 3 $F(x, u, c)$ is positive definite in $M(\eta)$ and $P_1(h) \subset M(\eta)$.

Q.E.D.

Theorem 2. If n satisfies the condition (37), then the equilibrium point a_1 of the system (1) is asymptotically stable in the large with respect to the set

$$P = \{(X, U, C): (K_3/10K_1)k \leq X,\ k \leq U,\ (K_3/2K_5)k \leq C\}, \tag{38}$$

in the sense of Definition 2.

Proof. The following conditions hold in $P_1(h)$, which is defined by (34):

1. every solution of (6) remains in $P_1(h)$ for all $t \geq t_0$, if the initial value belongs to $P_1(h)$ by Lemma 4 and transformation (5),
2. $P_1(h)$ is a compact set and $(0, 0, 0) \in P_1(h)$,
3. there is a continuously differentiable function $V(x, u, c)$ defined by (17) with properties
   a) $V(x, u, c)$ is positive definite in $P_1(h)$,
   b) $\dot{V}(x(t), u(t), c(t))|_{(6)} = -F(x, u, c)$ is negative definite in $P_1(h)$ by Lemma 5.

Then the equilibrium point (0, 0, 0) of the system (6) is stable with respect to the set $P_1(h)$ and all the solutions starting in $P_1(h)$ approach the origin (0, 0, 0) as $t \to \infty$ by the extended stability theorem of LaSalle [2]. Therefore, equilibrium point a_1 of (1) is asymptotically stable in the large with respect to $P(h)$ for all $h>L$, then to the set P.

Q.E.D.

Remark 3. Denote $N(n)=10(n+10)(n-1)/n(n+120)$, $1<n<485$. Then

$$\frac{dN(n)}{dn}=\frac{10(111n^2+2n+1200)}{n^2(n+120)^2}>0 \text{ for } 1<n<485.$$

Therefore, $N(n)$ is an increasing function of n for $1<n<485$, then

$$N(n)\leq N(4)<0.846 \text{ for } 1<n\leq 4. \tag{39}$$

On the other hand

$$\frac{dQ(n)}{dn}=\frac{-3025(q^{\frac{1}{2}}+2)(n-1)}{24q^2(q^{\frac{1}{2}}+1)^2(n+120)^3}<0 \text{ for } 1<n<485.$$

Therefore, $Q(n)$ is a decreasing function of n for $1<n<485$, then

$$Q(n)\geq Q(4)>0.851 \text{ for } 1<n\leq 4. \tag{40}$$

That is by (39), (40) $N(n)<Q(n)$ for $1<n\leq 4$. Therefore the condition (37) holds for $1<n\leq 4$ and we obtain

Corollary 2.1. If $K_1K_4<K_2K_3\leq 4K_1K_4$, then the equilibrium point a_1 of (1) is asymptotically stable in the large with respect to the set P defined by (38).

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Author's address:

Karol Bachratý, poslucháč MFF UK
Mlynská dolina
842 15 Bratislava

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РЕЗЮМЕ

ОБ УСТОЙЧИВОСТИ МОДЕЛИ РЕАКЦИИ БЕЛОУСОВА—ЖАБОТИНСКОГО

Карол Бахраты, Братислава

В работе рассматривается устойчивость системы (1), которая представляет модель Веисбуха—Саломона—Атлана реакции Белоусова—Жаботинского. Доказывается достаточное условие экспоненциальной устойчивости и достаточное условие асимптотической устайчивости в некотором множестве.

SÚHRN

O STABILITE MODELU BELOUSOVEJ—ŽABOTINSKÉHO REAKCIE

Karol Bachratý, Bratislava

V práci sa pojednáva o stabilite riešení diferenciálneho systému (1), ktorý predstavuje Weisbuchov—Salomonov—Atlanov model Belusovej—Žabotinského reakcie. Dokazuje sa postačujúca podmienka exponenciálnej stability a postačujúca podmienka asymptotickej stability vzhľadom na nijakú podmnožinu R^3.

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ON FUČÍK'S BOUNDARY VALUE PROBLEM $x''(t)+x(t)=\alpha(x(t))^-+p(t),\ x(0)=x(\pi)=0$

ZDENEK SCHNEIDER, Bratislava

1. Formulation of the problem

Consider the boundary value problem

$$x''(t)+x(t)=\alpha(x(t))^-+p(t),\ x(0)=x(\pi)=0 \tag{1.1}$$

where $(x(t))^-=\max\{-x(t), 0\}$, α is a positive constant and $p(t)\in C(\langle 0, \pi\rangle)$ is such that

$$\int_0^\pi p(t)\sin(t)\,\mathrm{d}t\leqq 0. \tag{1.2}$$

In the paper [3] S. Fučík poses the question as to whether (1.2) is a necessary and sufficient condition in order that (1.1) have a solution for any $\alpha>0$.

In the paper [1] L. Aquinaldo and K. Schmitt answered the question in the affirmative manner. They used Mawhin's coincidence theory [4].

The main difficulties in (1.1) are

a. the Green's function for associated linear problem

$$x''(t)+x(t)=0,\quad x(0)=x(\pi)=0$$

does not exist,

b. the nonlinear part of the problem (1.1)

$$\alpha(x(t))^-+p(t)$$

is a nondifferentiable function.

In this paper we use for solving the problem (1.1) Cesari's [2] alternative method. The foundation of this method may be described in this way.

Let us consider in a real separable Hilbert space S an operator equation $Ex=Nx$. Let $E\colon \mathcal{D}(E)\subset S\to S$ be a linear operator, let $N\colon \mathcal{D}(N)\subset S\to S$ be a nonlinear operator with $\mathscr{D}(N)\cap\mathscr{D}(E)\neq\emptyset$. Let $S=S_0\oplus S_1$ (direct sum). Let P, H be operators with the following properties:

P is a projection, $P\colon S\to S_0$, $\mathscr{R}(P)=S_0$, $\mathscr{N}(P)=S_1$, $Px\in\mathscr{D}(E)$ for $x\in S$,
H is a linear operator, $H\colon S_1\to S_1$, $H(I-P)Nx\in\mathscr{D}(E)$ for $x\in\mathscr{D}(N)$

and

$$\begin{aligned} &H(I-P)Ex=(I-P)x && x\in\mathscr{D}(E)\\ &PEx=EPx && x\in\mathscr{D}(E)\\ &EH(I-P)Nx=(I-P)Nx && x\in\mathscr{D}(N). \end{aligned} \tag{1.3}$$

Then the problem $Ex=Nx$ is equivalent to the system of two equations

$$x=Px+H(I-P)Nx \tag{1.4}$$

$$P(Ex-Nx)=0 \tag{1.5}$$

The equation (1.4) is said to be the auxiliary equation and (1.5) the bifurcation or determining equation. If S_0 is a finite dimensional space then (1.5) is a finite system of nonlinear equations in finite dimensional space.

The equation (1.4) in Cesari's papers is usually solved by Banach's fixed point theorem and the system (1.5) is analyzed by considerations based on the degree of a mapping [5].

The basic theorem for us is Theorem viii in [2] p. 398. Some hypotheses are given under which the problem $Ex=Nx$ has at least one solution.

We shall investigate under which conditions the hypotheses from Theorem viii are fulfilled.

Our results are not as general as in [1]. However, together with sufficient conditions for the existence of a solution $y(t)$ of the problem (1.1) we are able to give error bounds for $y(t)-x_0(t)$ and $y(t)-x_G(t)$ where $x_0(t)$ is given aproximation of the "solution" $y(t)$ and $x_G(t)$ is Galerkin's approximation for $y(t)$.

Clearly if $\int_0^\pi p(t)\sin(t)\,dt=0$ then using Fredholm's theorems we are able to argue that the solution $u_0(t)$ of the problem $x''(t)+x(t)=p(t)$, $x(0)=x(\pi)=0$ exists. The solution of this problem is also the function $u_0(t)+a.\sin(t)$ for arbitrary $a\in\mathbf{R}$. There exists a positive number A such that for $a>A$ holds $u_0(t)+a.\sin(t)\geqq 0$. This function is also a solution of the problem (1.1).

In what follows we will assume that $\int_0^\pi p(t)\sin(t)\,dt<0$.

2. Background

Let us denote $S=L^2(0,\pi)$. It is a real separable Hilbert space with norm $\|.\|$ and scalar product $(.,.)$ defined as usually: for $f, g\in S$ let be

$$(f, g)=\int_0^\pi f(t)g(t)\,\mathrm{d}t \text{ and } \|f\|=(f, f)^{1/2}.$$

Let $\varphi_j(t)=\sqrt{\frac{2}{\pi}}\sin(jt)$ $j=1, 2, \ldots,$ Obviously the system $\{\varphi_1(t), \ldots, \varphi_j(t), \ldots\}$ is a complete orthonormal system in S. Let us define the operator E as follows

$$\mathscr{D}(E)=\{x\in S\,|\,x(0)=x(\pi)=0,\ x(t)\in C^{(1)}(\langle 0, \pi\rangle),\ x''(t)\in S\}$$

$$E(x)(t)=x''(t)+x(t).$$

Obviously E: $\mathscr{D}(E)\subset S\to S$ and it is a selfadjoint (symmetric) operator and it holds

$$E\varphi_j(t)=-\lambda_j\varphi_j(t) \text{ with } \lambda_j=(j^2-1) \text{ for } j=1, 2, \ldots.$$

Further let N be the operator with

$$\mathscr{D}(N)=S$$

$$N(x)(t)=\alpha(x(t))^- + p(t).$$

Evidently N: $\mathscr{D}(N)\to S$ and it is easy to see that N is Lipschitzian with constant α i.e.

$$\|Nx_1-Nx_2\|\leqq\alpha\|x_1-x_2\| \text{ for } x_1, x_2\in S.$$

Inasmuch as $\mathscr{D}(E)\cap\mathscr{D}(N)\neq\emptyset$ we can the problem (1.1) rewrite in the form $Ex=Nx$.

Further let $S_0=\text{span}\,(\varphi_1(t))$ and $S_1=\text{span}\,(\varphi_2(t), \varphi_3(t), \ldots)$. Then $S_0\subset\mathscr{D}(E)$ and $S=S_0\oplus S_1$.

For $x\in S$ let

$$\sum_{j=1}^{\infty} a_j\varphi_j(t)$$

be associated Fourier series with $a_j=(x, \varphi_j)$.

Let us define the projection P as follows. If $x(t)=\sum_{j=1}^{\infty} a_j\varphi_j(t)$ then $P(x)(t)=a_1\varphi_1(t)$. It holds $\mathscr{R}(P)=S_0$, $\mathscr{N}(P)=S_1$.

Let us define the operator H: $S_1\to S_1$ in the following way. If $x\in S_1$ i.e. $x(t)=\sum_{j=2}^{\infty} a_j\varphi_j(t)$ then

$$H(x)(t)=-\sum_{j=2}^{\infty}\lambda_j^{-1}a_j\varphi_j(t).$$

Obviously H is a linear operator and $Hx \in S_1$.

Lemma 2.1. If $x \in S_1$ then $Hx \in \mathcal{D}(E)$.

Sketch of proof.

Let $u(t) = H(x)(t)$. It is easy to see (using Weierstrass's theorem) that $u(t) \in C^{(1)}(\langle 0, \pi \rangle)$ and $u(0) = u(\pi) = 0$. The function $v(t) = -\sum_{j=2}^{\infty} \lambda_j^{-1} a_j \varphi_j(t)$ belongs to $L^2(0, \pi) \subset L(0, \pi)$. Finally, we can show that for $0 \leqq t \leqq \pi$

$u'(t) = u'(0) + \int_0^t v(s)\, ds$, thus $u''(t)$ exists a.e. and belongs to S.

Q.E.D.

Lemma 2.2. It holds

$$\begin{aligned} &H(I-P)Ex = (I-P)x && x \in \mathcal{D}(E) \\ &PEx = EPx && x \in \mathcal{D}(E) \\ &EH(I-P)Nx = (I-P)Nx && x \in \mathcal{D}(N). \end{aligned}$$

Proof.

By direct computation.

Following [2] the equation $Ex = Nx$ is equivalent to the system of two equations

$$x = Px + H(I-P)Nx, \quad P(Ex - Nx) = 0.$$

3. Auxiliary equation

Let us denote $x_0(t) = c_0 \varphi_1(t)$ for an arbitrary, but fixed $x_0 \in \mathbf{R}$. For a bounded function $x(t) \in S$ let $\mu(x) = \sup_{\langle 0, \pi \rangle} |x(t)|$. Let us denote $\Delta = H(I-P)(Ex_0 - Nx_0)$. Then $\Delta = -H(I-P)p(t) = \sum_{j=2}^{\infty} \lambda_j^{-1} p_j \varphi_j(t)$ with $p_j = (p(t), \varphi_j(t))$ $j = 1, 2, \ldots$.

Obviously there exist numbers b, b' such that

$$\|\Delta\| \leqq b, \quad \mu(\Delta) \leqq b' \tag{3.1}$$

holds.

If $x \in S$ then it is possible to show that it holds

$$\|H(I-P)x\| \leqq k\|x\| \text{ and } \mu(H(I-P)x) \leqq k'\|x\|$$
$$\text{with } k = 1/3 \text{ and } k' = 0{,}29312571 \tag{3.2}$$

Using Theorem iv [2] p. 393 if there exist numbers c, d, r, R_0, α such that

1. $0 < c < d$, $0 < r < R_0$

2. $k\alpha < 1$

3. $k\alpha d < d - c - b$ (3.3)

4. $k'\alpha d < R_0 - r - b'$

5. the implication $z \in S_0$, $\|z\| \leqq c \Rightarrow \mu(z) \leqq r$ holds then $Px + H(I-P)Nx$ is a contractive mapping on complete subspace

$$S_0^* = \{x \in S \mid Px = Px^*, \|x - x_0\| \leqq d, \mu(x - x_0) \leqq R_0\}$$

for arbitrary but fixed $x^* \in V = \{x \in S_0 \mid \|x - x_0\| \leqq c\}$.

***Lemma* 3. 1.** *If*

$$\alpha < 3 \tag{3.4}$$

and for arbitrary $c > 0$ it is

$$d > (c + b)/(1 - k\alpha) \tag{3.5}$$

$$r = \sqrt{\frac{2}{\pi}} \cdot c \tag{3.6}$$

$$R_0 > k'\alpha d + r + b' \tag{3.7}$$

then the conditions (3.3) are fulfilled.

So if the conditions (3.4)—(3.7) are satisfied, the equation $x = Px + H(I-P)Nx$ has exactly one solution $x(x^*) \in S_0^*$. This solution depends continuously on $x^* \in V$. From Lemma 2.1 we see that $x(x^*) \in \mathscr{D}(E) \cap \mathscr{D}(N)$ and so $x(x^*)$ will be the solution of the problem $Ex = Nx$ if and only if $P(E(x(x^*)) - N(x(x^*))) = 0$.

4. Determining equation

The solvability of the equation $P(E(x(x^*)) - N(x(x^*))) = 0$ is analyzed in [2] by considerations based on the degree of a mapping.

In the sense of previous notations (see part 3) the condition (1.2) may be rewritten in the form $p_1 < 0$.

Let us define the following notation: for given c_0, $c > 0$ let

$$Q = \min |(E(c_1\varphi_1) - N(c_1\varphi_1), \varphi_1)| \tag{4.1}$$

where the minimum is taken over the set $c_1 \in \{c_0 + c, c_0 - c\}$.

Following Theorem viii [2] p. 397 sufficient conditions for the existence of solution $y(t)$ of the problem (1.1) are: for given $c_0 \in \mathbf{R}$

1. there exist numbers c, d, r, R_0, α such that the conditions (3.3) are satisfied

2. $(-p_1 - \alpha(c_0 + c)^-) . (-p_1 - \alpha(c_0 - c)^-) < 0$ (4.2)

3. $\alpha(k\alpha d + b) < Q$. (4.3)

Lemma 4.1. The condition

$$(-p_1-\alpha(c_0+c)^-).(-p_1-\alpha(c_0-c)^-)<0$$

is, for given c_0, satisfied in the following cases:

$$(c_0>0)\wedge\left(c>-\frac{1}{\alpha}p_1+c_0\right) \tag{4.4}$$

$$(c_0=0)\wedge\left(c>-\frac{1}{\alpha}p_1\right) \tag{4.5}$$

$$\left(\frac{1}{2}\frac{1}{\alpha}p_1<c_0<0\right)\wedge\left(-\frac{1}{\alpha}p_1+c_0<c\right) \tag{4.6}$$

$$\left(c_0=\frac{1}{2}\frac{1}{\alpha}p_1\right)\wedge(-c_0<c) \tag{4.7}$$

$$\left(\frac{1}{\alpha}p_1<c_0<\frac{1}{2}\frac{1}{\alpha}p_1\right)\wedge\left(-\frac{1}{\alpha}p_1+c_0<c<-c_0\right) \tag{4.8}$$

$$\left(c_0=\frac{1}{\alpha}p_1\right)\wedge(0<c<-c_0) \tag{4.9}$$

$$\left(c_0<\frac{1}{\alpha}p_1\right)\wedge\left(\frac{1}{\alpha}p_1-c_0<c<-c_0\right) \tag{4.10}$$

$$\left(c_0<\frac{1}{2}\frac{1}{\alpha}p_1\right)\wedge(-c_0<c) \tag{4.11}$$

$$\left(c_0<\frac{1}{2}\frac{1}{\alpha}p_1\right)\wedge(-c_0=c). \tag{4.12}$$

Proof.

By direct computation.

Lemma 4.2. For the value Q (from (4.1)) it holds

1. if $(c_0>0)$ and
   $(0<c<c_0)$ then $Q=-p_1$
   $(c=c_0)$ then $Q=-p_1$
   $\left(c_0<c<-\frac{1}{\alpha}p_1+c_0\right)$ then $Q=-p_1-\alpha(c-c_0)$
   $\left(c=-\frac{1}{\alpha}p_1+c_0\right)$ then $Q=0$
   $\left(-\frac{1}{\alpha}p_1+c_0<c<-\frac{1}{\alpha}2p_1+c_0\right)$ then $Q=p_1+\alpha(c-c_0)$ (4.13)
   $\left(-\frac{1}{\alpha}2p_1+c_0=c\right)$ then $Q=-p_1$ (4.14)

$\left(-\frac{1}{\alpha}2p_1+c_0<c\right)$ then $Q=-p_1$ (4.15)

2. if $(c_0=0)$ and

$\left(0<c<-\frac{1}{\alpha}p_1\right)$ then $Q=-p_1-\alpha c$

$\left(c=-\frac{1}{\alpha}p_1\right)$ then $Q=0$

$\left(-\frac{1}{\alpha}p_1<c<-\frac{1}{\alpha}2p_1\right)$ then $Q=p_1+\alpha c$ (4.16)

$\left(c=-\frac{1}{\alpha}2p_1\right)$ then $Q=-p_1$ (4.17)

$\left(-\frac{1}{\alpha}2p_1<c\right)$ then $Q=-p_1$ (4.18)

3*a*. if $\left(\frac{1}{\alpha}\frac{1}{2}p_1<c_0<0\right)$ and

$(0<c<-c_0)$ then $Q=-p_1-\alpha(-c_0+c)$

$(c=-c_0)$ then $Q=-p_1-\alpha(-2c_0)$

$\left(-c_0<c<-\frac{1}{\alpha}p_1+c_0\right)$ then $Q=-p_1-\alpha(c-c_0)$

$\left(c=-\frac{1}{\alpha}p_1+c_0\right)$ then $Q=0$

$\left(-\frac{1}{\alpha}p_1+c_0<c<-\frac{1}{\alpha}2p_1+c_0\right)$ then $Q=p_1+\alpha(c-c_0)$ (4.19)

$\left(c=-\frac{1}{\alpha}2p_1+c_0\right)$ then $Q=-1$ (4.20)

$\left(-\frac{1}{\alpha}2p_1+c_0<c\right)$ then $Q=-p_1$ (4.21)

3b. if $\left(c_0=\frac{1}{\alpha}\frac{1}{2}p_1\right)$ and

$(0<c<-c_0)$ then $Q=-p_1-\alpha(-c_0+c)$

$(c=-c_0)$ then $Q=0$

$\left(-c_0<c<-\frac{1}{\alpha}\frac{3}{2}p_1\right)$ then $Q=p_1+\alpha(c-c_0)$ (4.22)

$\left(c=-\frac{1}{\alpha}\frac{3}{2}p_1\right)$ then $Q=-p_1$ (4.23)

$\left(-\frac{1}{\alpha}\frac{3}{2}p_1<c\right)$ then $Q=-p_1$ (4.24)

3c. if $\left(\frac{1}{\alpha}p_1<c_0<\frac{1}{\alpha}\frac{1}{2}p_1\right)$ and

$$\left(0<c<-\frac{1}{\alpha}p_1+c_0\right) \text{ then } Q=-p_1-\alpha(-c_0+c)$$

$$\left(c=-\frac{1}{\alpha}p_1+c_0\right) \text{ then } Q=0$$

$$\left(-\frac{1}{\alpha}p_1+c_0<c<-c_0\right) \text{ then } Q=p_1+\alpha(-c_0+c) \tag{4.25}$$

$$(c=-c_0) \text{ then } Q=p_1+\alpha(-2c_0) \tag{4.26}$$

$$\left(-c_0<c<-\frac{1}{\alpha}2p_1+c_0\right) \text{ then } Q=p_1+\alpha(c-c_0) \tag{4.27}$$

$$\left(c=-\frac{1}{\alpha}2p_1+c_0\right) \text{ then } Q=-p_1 \tag{4.28}$$

$$\left(-\frac{1}{\alpha}2p_1+c_0<c\right) \text{ then } Q=-p_1 \tag{4.29}$$

3d. if $\left(c_0=\frac{1}{\alpha}p_1\right)$ and

$$(0<c<-c_0) \text{ then } Q=-p_1-\alpha(-c_0-c) \tag{4.30}$$

$$(c=-c_0))\text{then } Q=-p_1 \tag{4.31}$$

$$(-c_0<c) \text{ then } Q=-p_1 \tag{4.32}$$

3e. if $\left(c_0<\frac{1}{\alpha}p_1\right)$ and

$$\left(0<c<\frac{1}{\alpha}p_1-c_0\right) \text{ then } Q=p_1+\alpha(-c_0-c)$$

$$\left(c=\frac{1}{\alpha}p_1-c_0\right) \text{ then } Q=0$$

$$\left(\frac{1}{\alpha}p_1-c_0<c<-c_0\right) \text{ then } Q=-p_1-\alpha(-c_0-c) \tag{4.33}$$

$$(c=-c_0) \text{ then } Q=-p_1 \tag{4.34}$$

$$(-c_0<c) \text{ then } Q=-p_1 \tag{4.35}$$

Proof.
By direct computation.

5. Conclusion

In the sense of part 3 the number $c_0 \in \mathbf{R}$ is arbitrary but fixed. The function $x_0(t)=c_0\varphi_1(t)$ is given approximation for "solution" $y(t)$. If for given c_0 the conditions from Lemma 3.1 and from (4.2) and (4.3) are fulfilled then there exists a solution $y(t)$.

The choice $c_0=\frac{1}{\alpha}p_1$ implies that $x_0(t)$ is the first Galerkin's approximation for $y(t)$.

From Lemma 3.1, Lemma 4.1, and Lemma 4.2 we obtain existence and approximation theorems. Some of them are

Theorem 5.1. (on Galerkin's approximation) Let

$$k\alpha < 1/2 \tag{5.1}$$

$$-p_1 > \left(\left(\frac{1-k\alpha}{1-2k\alpha}\right)\cdot\left(\alpha b + \frac{k\alpha^2}{1-k\alpha}\,b\right)\right) \tag{5.2}$$

then for $x_0(t) = \frac{1}{\alpha}\,p_1\varphi_1(t)$ and $c = -\frac{1}{\alpha}\,p_1$ there exists a solution $y(t)$ of the problem (1.1) and it holds

$$\|y - x_0\| \leqq d,\ \ \|Py - x_0\| \leqq c,\ \ \mu(y - x_0) \leqq R_0,\ \ \mu(Py - x_0) \leqq r.$$

Remark. The numbers b, b' are determined by (3.1), $k = 1/3$, $k' =$ 0,29312571 by (3.2), the numbers d, r, R_0 are determined by (3.4), (3.5), (3.6), (3.7).

Proof of Theorem.

From Lemma 4.1 it follows that for $c_0 = \frac{1}{\alpha}\,p_1 < \frac{1}{2}\frac{1}{\alpha}\,p_1$ is $(-p_1 - \alpha(c_0 + c)^-)$ $\cdot\ (-p_1 - \alpha(c_0 - c)^-) < 0$ if $c = -c_0 = -\frac{1}{\alpha}\,p_1$. (The condition (4.12)). From Lemma 4.2 $Q = -p_1$ (The condition (4.31)). From Lemma 3.1 it follows that necessarily $\alpha < 3$ and for $c = -\frac{1}{\alpha}\,p_1$ the relations

$$d > \frac{c+b}{1-k\alpha},\ \ r = \sqrt{\frac{2}{\pi}}\,c,\ \ R_0 > k'\alpha d + r + b'$$

must be satisfied.

From condition (4.3) we have

$$d < \frac{-p_1 - \alpha b}{k\alpha^2}$$

and so necessarily

$$\frac{-\frac{1}{\alpha}\,p_1 + b}{1-k\alpha} < \frac{-p_1 - \alpha b}{k\alpha^2}.$$

Hence

$$k\alpha < 1/2$$

$$-p_1 > \left(\left(\frac{1-k\alpha}{1-2k\alpha}\right)\cdot\left(\alpha b + \frac{k\alpha^2}{1-k\alpha}\,b\right)\right) \tag{Q.E.D.}$$

Similarly we can prove

Theorem 5.2. ($c_0>0$) Let

$$k\alpha<1/3, \tag{5.3}$$

let there exist a number $c_0>0$ such that

$$-p_1>\left(\left(\frac{1-k\alpha}{1-3k\alpha}\right)\cdot\left(\alpha b+\frac{k\alpha^2b^2}{1-k\alpha}\right)\right)+\left(\frac{k\alpha^2}{1-3k\alpha}\right)c_0. \tag{5.4}$$

Then for $x_0(t)=c_0\varphi_1(t)$ and $c=c_0-\frac{1}{\alpha}2p_1$ there exists a solution $y(t)$ of the problem (1.1) and it holds

$$\|y-x_0\|\leqq d,\ \|Py-x_0\|\leqq c,\ \mu(y-x_0)\leqq R_0,\ \mu(Py-x_0)\leqq r.$$

Theorem 5.3. ($c_0=0$) Let

$$k\alpha<1/3, \tag{5.5}$$

and

$$-p_1>\left(\left(\frac{1-k\alpha}{1-3k\alpha}\right)\cdot\left(\alpha b+\frac{k\alpha^2}{1-k\alpha}b\right)\right). \tag{5.6}$$

Then for $x_0(t)=0$ and $c=-\frac{1}{\alpha}2p_1$ there exists a solution $y(t)$ of the problem (1.1) and it holds

$$\|y\|\leqq d,\ \|Py\|\leqq c,\ \mu(y)\leqq R_0,\ \mu(Py)\leqq r.$$

Theorem 5.4. (on Galerkin's approximation) Let

$$k\alpha<1/2, \tag{5.7}$$

let there exist δ: $0<\delta<-\frac{1}{\alpha}p_1$ such that

$$\delta>\left(\frac{1}{\alpha}\left(\frac{1-k\alpha}{1-2k\alpha}\right)\cdot\left(\alpha b+\frac{k\alpha^2}{1-k\alpha}b\right)\right). \tag{5.8}$$

Then for $x_0(t)=\frac{1}{\alpha}p_1\varphi_1(t)$ and $c=\delta$ there exists a solution $y(t)$ of the problem (1.1) and it holds

$$\|y-x_0\|\leqq d,\ \|Py-x_0\|\leqq c,\ \mu(y-x_0)\leqq R_0,\ \mu(Py-x_0)\leqq r.$$

Theorem 5.5. Let

$$k\alpha=1/2, \tag{5.9}$$

let there exist ω: $0<\omega<-\frac{1}{\alpha}\frac{1}{2}p_1$ such that

$$-\alpha\omega(1-3k\alpha)>\left((1-k\alpha).\left(\alpha b+\frac{k\alpha^2}{1-k\alpha}b\right)\right). \tag{5.10}$$

Then for $x_0(t)=\left(\frac{1}{\alpha}p_1+\omega\right)\varphi_1(t)$ and $c=-\frac{1}{\alpha}p_1-\omega$ there exists a solution $y(t)$ of the problem (1.1) and it holds

$$\|y-x_0\|\leqq d,\ \|Py-x_0\|\leqq c,\ \mu(y-x_0)\leqq R_0,\ \mu(Py-x_0)\leqq r.$$

Theorem 5.6. Let

$$1/2<k\alpha<1, \tag{5.11}$$

exists numbers ω, δ:

$$0<\omega<-\frac{1}{\alpha}\frac{1}{2}p_1 \tag{5.12}$$

$$0<\delta<-\frac{1}{\alpha}p_1 \tag{5.13}$$

such that

$$-\alpha\omega(1-3k\alpha)>\left((1-k\alpha)\cdot\left(\alpha b+\frac{k\alpha^2}{1-k\alpha}b\right)\right)-(1-2k\alpha)\cdot(-p_1+\alpha\delta). \tag{5.14}$$

Then for $x_0(t)=\left(\frac{1}{\alpha}p_1+\omega\right)\varphi_1(t)$ and $c=-\frac{1}{\alpha}p_1-\omega+\delta$ there exists a solution $y(t)$ of the problem (1.1) and it holds

$$\|y-x_0\|\leqq d,\ \|Py-x_0\|\leqq c,\ \mu(y-x_0)\leqq R_0,\ \mu(Py-x_0)\leqq r.$$

6. Example

Let us consider the nonlinear boundary value problem

$$x''(t)+x(t)=1{,}2(x(t))^-+p(t),\ x(o)=x(\pi)=0 \tag{6.1}$$

with $p(t)=-\varphi_1(t)+\varphi_3(t)+\varphi_4(t)+\varphi_5(t)$.

Because $k\alpha<1/2$, we can use the theorems 5.1 and 5.4. Then $p_1=-1$ and $x_0(t)=-0{,}6649038\sin(t)$. From (3.1) we obtain that $b=0{,}147667042$ and $b'=0{,}1861173063$.

Following Theorem 5.1 we see that inequalities (5.1) and (5.2) are fulfilled. Next for $c=0{,}83333333$ we obtain from Lemma 3.1

$d = 1{,}635001$, $r = 0{,}6649038$ and $R_0 = 1{,}42619$. So we that a solution $y(t)$ of the problem (6.1) exists and it holds

$$\begin{gathered} \|y(t) - (-0{,}6649038 \sin(t))\| \leqq 1{,}635001 \\ \|Py(t) - (-0{,}6649038 \sin(t))\| \leqq 0{,}8333333 \\ \sup_{\langle 0, \pi \rangle} |y(t) - (-0{,}6649038 \sin(t)| \leqq 1{,}42619 \\ \sup_{\langle 0, \pi \rangle} |Py(t) - (-0{,}6649038 \sin(t))| \leqq 0{,}6649038. \end{gathered} \tag{6.2}$$

Following Theorem 5.4 we have to choose $0{,}73833521 < \delta < 0{,}8333333$. For $\delta = 0{,}7384$ *it is* $c = 0{,}7384$ and from Lemma 3.1 $d = 1{,}47678$, $r = 0{,}589158$ and $R_0 = 1{,}29479$. So we have that a solution of the problem (6.1) exists an it holds

$$\begin{gathered} \|y(t) - (-0{,}6649038 \, sin \, (t))\| \leqq 1{,}47678 \\ \|Py(t) - (-0{,}6649038 \, sin \, (t)\| \leqq 0{,}7384 \\ \sup_{\langle 0, \pi \rangle} |y(t) - (-0{,}6649038 \sin(t))| \leqq 1{,}29479 \\ \sup_{\langle 0, \pi \rangle} |Py(t) - (-0{,}6649038 \sin(t))| \leqq 0{,}5891158. \end{gathered} \tag{6.3}$$

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Author's address:

Received: 22. 4. 1981

Zdenek Schneider
Katedra numerickej matematiky MFF UK,
Mlynská dolina,
842 15 Bratislava.

SÚHRN

O FUČÍKOVEJ OKRAJOVEJ ÚLOHE
$x''(t)+x(t)=\alpha(x(t))+p(t),\ x(0)=x(\pi)=0$

Zdenek Schneider, Bratislava

V práci sú uvedené postačujúce podmienky na existenciu riešenia $y(t)$ okrajovej úlohy (1.1). Výsledky nie sú také všeobecné ako v práci [1], ale je uvedený odhad chyby rozdielu $y(t)-x_0(t)$, $y(t)-x_G(t)$ kde $x_0(t)$ je daná aproximácia riešenia a $x_G(t)$ je Galerkinovská aproximácia riešenia.

РЕЗЮМЕ

ОБ КРАЕВОЙ ЗАДАЧЕ ФУЧИКА
$x''(t)+x(t)=\alpha(x(t))+p(t),\ x(0)=x(\pi)=0$

Зденек Шнайдер, Братислава

В работе даны достаточные условия существования решения $y(t)$ краевой задачи (1.1). Наши достижения не так общие как в работе [1], но дана возможность оценить ошибку для $y(t)-x_0(t)$, $y(t)-x_G(t)$ где $x_0(t)$ данное приближение и $x_G(t)$ приближение по Галеркину решения $y(t)$.