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AN ITERATION METHOD FOR INITIAL-VALUE PROBLEMS OF RETARDED DIFFERENTIAL EQUATIONS

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1. Introduction

We consider a system of first order of retarded ordinary differential equations

$$x'(t)=f(t, x(t), x(\alpha(t))) \text{ for } 0\leqq t\leqq T \tag{1.1}$$

where f: $[0, T]\times R^n\times R^n\to R^n$ and $\alpha(t)\leqq t$.

Let $\bar{\tau}=\inf \alpha(t)$ for $0\leqq t\leqq T$. The set of t such that $\bar{\tau}\leqq t\leqq 0$ is called the "initial set". We assume that the initial condition is given on the initial set. If $\bar{\tau}<0$, then the initial condition is given on an interval; this contrasts with the ordinary differential equation case in which an initial condition at one point always suffices. Thus the initial condition is:

$$x(t)=\varphi(t) \text{ for } \bar{\tau}\leqq t\leqq 0 \tag{1.2}$$

for a given function $\varphi(t)\in C([\bar{\tau}, 0]; R^n)$. The function $\alpha(t)$ is usually called the retardation or lag function. It is assumed that $\alpha(t)$ is continuous for $t\in[0, T]$.

A indication of the importance of retarded differential equations is evidenced by the many different areas in which they describe physical systems, such as electrostatic charge problems, automatic controls, machine tools and biological systems. Algorithms for the numerical solution of (1.1), (1.2) have been proposed by El'sgol'ts [3], Cryer and Tavernini [1], Feldstein [4], Feldstein and Goodman [5], Hutchison [7], Tavernini [8], [9], Zverkina [12], [13]. All of these algorithms reduce to one step methods or to linear multistep methods if there is no retardation. The stability of linear multistep methods for retarded differential equations has been considered by Cryer [2] and Wiederholt [10], [11]. Wiederholt shows that the stability regions for retarded differential equations are significantly different from the stability regions for ordinary differential equations.

In the present paper we describe an iteration method for solving the initial-value problem (1.1), (1.2). In this method the problem (1.1), (1.2) is reduced to a sequence of initial-value problems for a system of ordinary differential equations. These systems can be solved using numerical methods for ordinary differential equations.

2. The iteration method

We shall require that the following assumptions are satisfied:

A1: The scalar function $\Psi(t, u, v)$ is positive and continuous on $[0, T]$ and for $0 \leqq u, v \leqq 2r$.

A2: $\Psi(t, u, v)$ is nondecreasing in v for fixed (t, u) and the initial-value problem

$$u'(t) = \Psi(t, u, u), \quad 0 \leqq t \leqq T,$$
$$u(0) = 0$$

has only the trivial solution $u(t) \equiv 0$.

A3: The initial-value problem

$$u'(t) = \Psi(t, u(t), \sigma(t)), \quad t \in [0, T]$$
$$u(0) = 0$$

has a unique solution $u(t)$, $0 \leqq u(t) \leqq 2r$ which any be found for each fixed continuous function $\sigma(t)$, $\sigma(t) \in [0, 2r]$.

A4: Let the function $f(t, x, y)$, $x = (x_1, x_2, \ldots, x_n)^T$,

$$y = (y_1, y_2, \ldots, y_n)^T,$$

be continuous in

$$G := \{(t, x, y) | 0 \leqq t \leqq T, |x_i| \leqq r, |y_i| \leqq r, i = 1, 2, \ldots, n\}$$

and let

$$\|f(t, x, y) - f(t, x^*, y^*)\|_\infty \leqq \Psi(t, \|x - x^*\|_\infty, \|y - y^*\|_\infty)$$

for all (t, x, y); $(t, x^*, y^*) \in G$.

A5: The initial-value problem

$$x'(t) = f(t, x(t), y(t))$$
$$x(0) = \varphi(0)$$

has a unique solution $x(t)$, $|x_i(t)| \leqq r$, $i = 1, 2, \ldots, n$, which may be found for each fixed continuous function $y(t)$ with $|y_i(t)| \leqq r$, $i = 1, 2, \ldots, n$.

Theorem: Let the assumptions A1—A5 be fulfilled. Then the initial-value problem (1.1), (1.2) has a unique solution which may be found with help of the iteration method:

$$
\begin{aligned}
x'_{n+1}(t) &= f(t, x_{n+1}(t), x_n(\alpha(t))) \\
x_{n+1}(t) &= \varphi(t) \text{ for } t \in [\bar{\tau}, 0] \qquad n = 0, 1, \ldots, \\
x_0(t) &= \varphi(t) \text{ for } t \in [\bar{\tau}, 0],\ x_0(t) = \varphi(0) \text{ for } [0, T].
\end{aligned} \tag{2.1}
$$

The convergence of the sequence $\{x_n(t)\}$ on $[0,T]$ is given by

$$\|x_n(t) - x(t)\|_\infty \leqq \varepsilon_n(t), \qquad n = 0, 1, \ldots. \tag{2.2}$$

where the sequence $\{\varepsilon_n(t)\}$ satisfying the initial-value problems

$$
\begin{aligned}
\varepsilon'_{n+1}(t) &= \Psi(t, \varepsilon_{n+1}(t), \varepsilon_n(\alpha(t))), \qquad t \in [0, T] \\
\varepsilon_{n+1}(0) &= 0, \qquad n = 0, 1, \ldots,
\end{aligned} \tag{2.3}
$$

converges uniformly to zero and $\varepsilon_0(t) \equiv 2r$; $\varepsilon_{n+1}(t) = 0$ for $[\bar{\tau}, 0]$.

Proof:

a) The uniform convergence of the sequence $\{\varepsilon_n(t)\}$ defined by the recurrence relation (2.3) must be proved. Let $\varepsilon_n(t)$ be a unique solution of the problem

$$
\begin{aligned}
\varepsilon'_n(t) &= \Psi(t, \varepsilon_n(t), \varepsilon_{n-1}(\alpha(t))) \\
\varepsilon_n(t) &= 0 \text{ for } t \in [\bar{\tau}, 0].
\end{aligned} \tag{2.4}
$$

From (2.3) under the induction hypotheses that $\varepsilon_n(t) \leqq \varepsilon_{n-1}(t)$ and from the assumptions A1 and A2 follows that

$$0 \leqq \varepsilon'_{n+1}(t) \leqq \Psi(t, \varepsilon_{n+1}(t), \varepsilon_{n-1}(\alpha(t)))$$

which implies

$$0 \leqq \varepsilon_{n+1}(t) \leqq \int_0^t \Psi(s, \varepsilon_{n+1}(s), \varepsilon_{n-1}(\alpha(s)))\, ds \text{ for } t \in [0, T],$$

$$\varepsilon_0(t) \equiv 2r, \qquad n = 1, 2, \ldots.$$

From (2.4) we obtain $\varepsilon_{n+1}(t) \leqq \varepsilon_n(t)$ directly.

Hence, the sequence $\{\varepsilon_n(t)\}$, $n = 0, 1, \ldots$, is nonincreasing for each $t \in [0, T]$. Then from $\varepsilon_n(t) \geqq 0$ and from the assumption A2 we obtain that the limit of the sequence $\{\varepsilon_n(t)\}$ must be zero. Now, the uniform convergence to zero must be proved. From (2.4) we obtain

$$\lim_{n\to\infty} \varepsilon_n(t) = \lim_{n\to\infty} \int_0^t \Psi(s, \varepsilon_n(s), \varepsilon_{n-1}(\alpha(s)))\, ds = \varepsilon(t)$$

where $\varepsilon(t)$ is a solution of the problem as in A2. Therefore we have $\lim \varepsilon_n(t) = 0$ for $n \to \infty$, while $\varepsilon(t) \equiv 0$. From Dini's theorem (see Günther/Beyer [6]) we can assert that the sequence $\{\varepsilon_n(t)\}$ converges uniformly as well as monotonously to zero on $[0, T]$.

b) The existence and uniqueness of the solution of the initial-value problem (1.1), (1.2) as a limit of the sequence $\{x_n(t)\}$ described by the recurrence realation (2.1) or

$$x_n(t) = \begin{cases} \int_0^t f(s, x_n(s), x_{n-1}(\alpha(s)))\,ds \text{ on } [0, T] \\ \varphi(t) \text{ for } [\bar{\tau}, 0] \end{cases}, \quad n = 1, 2, \ldots$$

with $x_0(t) = \varphi(t)$ for $t \in [\bar{\tau}, 0]$, $x_0(t) = \varphi(0)$ for $t \in [0, T]$ must be proved, The sufficient condition for this is the proof of fundamentality of the sequence $\{x_n(t)\}$ on $[0, T]$.

Let us now indicate

$$u_{n,m}(t) = \|x_n(t) - x_m(t)\|_\infty, \qquad m > n,$$

then we have

$$u_{n,m}(t) = \left\| \int_0^t [f(s, x_n(s), x_{n-1}(\alpha(s))) - f(s, x_m(s), x_{m-1}(\alpha(s)))]\,ds \right\|_\infty$$

$$\leqq \int_0^t \|f(s, x_n(s), x_{n-1}(\alpha(s))) - f(s, x_m(s), x_{m-1}(\alpha(s)))\|_\infty\,ds\,.$$

With the assumption A4 we obtain

$$u_{n,m}(t) \leqq \int_0^t \Psi(s, \|x_n(s) - x_m(s)\|_\infty, \|x_{n-1}(\alpha(s)) - x_{m-1}(\alpha(s))\|_\infty)\,ds$$

$$= \int_0^t \Psi(s, u_{n,m}(s), u_{n-1,m-1}(\alpha(s)))\,ds\,. \tag{2.5}$$

Let us give

$$u_{n-1,k}(t) \leqq \varepsilon_{n-1}(t), \qquad k > n-1,$$

then (2.5) under the assumption A2 implies

$$u_{n,m}(t) \leqq \int_0^t \Psi(s, u_{n,m}(s), \varepsilon_{n-1}(\alpha(s)))\,ds\,.$$

Now, under the assumption that $\varepsilon_n(t)$ is an unique solution of the problem

$$\begin{aligned} u'(t) &= \Psi(t, u(t), \varepsilon_{n-1}(\alpha(t))) \\ u(0) &= 0 \end{aligned}$$

we have

$$0 \leqq u_{n,m}(t) \leqq \varepsilon_n(t) \text{ for each } m > n\,. \tag{2.6}$$

Since $\varepsilon_{n+1}(t) \leqq \varepsilon_n(t)$, $n = 0, 1, \ldots$, and $\lim \varepsilon_n(t) = 0$ for $n \to \infty$, then $\varepsilon_n(t) \leqq \varepsilon^*$ for each $n > N$ and the relation (2.6) imply the fundamentality of the sequence $\{x_n(t)\}$ on $[0, T]$. If $m \to \infty$ then the limit of (2.6) is (2.2). Let $x(t)$ and $y(t)$ be two different solutions of (1.1), (1.2). Easily can be proved that

$$\|x_n(t)-y(t)\|_\infty \leqq \varepsilon_n(t).$$

Then

$$\|x(t)-y(t)\|_\infty = \|x(t)-x_n(t)\|_\infty + \|x_n(t)-y(t)\|_\infty \leqq 2 \, . \, \varepsilon_n(t).$$

After limiting for $n \to \infty$ we have $x(t) \equiv y(t)$.

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SÚHRN

ITERAČNÁ METÓDA PRE ZAČIATOČNÚ ÚLOHU DIFERENCIÁLNYCH ROVNÍC S ONESKORENÝM ARGUMENTOM

Karl Strehmel, Pavol Chocholatý

V práci sa uvádza iteračná metóda pre numerické riešenie diferenciálnych rovníc s oneskoreným argumentom. Sústava diferenciálnych rovníc s oneskorením sa pritom redukuje na postupnosť diferenciálnych rovníc bez oneskorenia. Stanovené sú postačujúce podmienky existencie a jednoznačnosti riešenia začiatočnej úlohy s oneskorením.

РЕЗЮМЕ

ИТЕРАЦИОННЫЙ МЕТОД ДЛЯ ЗАДАЧИ КОШИ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАЗДЫВАНИЕМ

Карл Штремел, Павол Хохолаты

В работе рассматривается итерационный метод для численного решения дифференциальных уравнений с запаздыванием. Система дифференциальных уравнений с запаздыванием сводится к посследовательности дифференциальных уравнений без запаздывания. Устанавливаются достаточные условия для существования и единственности решения задачи Коши с запаздыванием.