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A LOCAL ERROR ESTIMATION OF THE FORMULAS OF THE RUNGE—KUTTA—HUŤA TYPE OF THE FIFTH AND SIXTH ORDER FOR THE SYSTEM OF DIFFERENTIAL EQUATIONS

ANNA VALKOVÁ, Bratislava

In this paper are stated the error estimations of the approximate solution of the initial problem for the system of differential equations or for one equation, by the methods of Runge—Kutta type of the fifth and sixth order [1].

Our approach is analogical to the approach used in [2] for the estimations of the classical formulas of Runge—Kutta type of the fourth order.

Note: In this paper we shall refer to some relations which are stated in [3], [4]. For the reason of the large range of these relations we will not introduce them here again. In each case we shall only refer to the number of the relation stated in the papers [3] or [4]. To avoid misunderstanding, the relations stated in paper [3] will be numbered as follows: (1), (2), ..., (18). The relations defined in paper [4] will be numbered: (I.1), (I.2), ..., (I.18). The relations defined in this paper will be numbered: (II.1), (II.2), ... In the following text we shall use these abbreviations for the formulas: RKH5 — Huťa formula of the fifth order, RKH6 — Huťa formula of the sixth order, RKN5 — Nyström formula of the fifth order.

In paper [3] was stated the theoretical formula (18) for the local error of the approximate solution of the system (1), (2) by means of RKH5.

$$\omega_{n,r+1} = \max_{\xi \in \langle x_r, x_{r+1} \rangle} \sum_{i=1}^{n} \left| A_i (\xi - x_r)^6 + B_i \frac{(\xi - x_r)^7}{7!} \right| \tag{II.1}$$

where A_i, B_i, the coefficients at $(\xi - x_r)^6$ and $\frac{(\xi - x_r)^7}{7!}$ are described in formula (18). The theoretical formula (18) is awkward for computing the error and therefore we must be satisfied with its estimation only.

Similarly to [2] we shall execute the estimation under these three assumptions:

Lotkin's assumption:

$$|f_i| \leqq M, \quad \left| \frac{\partial^{l+j_1+j_2+\ldots+j_n} f_i}{\partial_x^l \partial y_1^{j_1} \ldots \partial y_n^{j_n}} \right| \leqq \frac{L_A^{l+j_1+j_2+\ldots+j_n}}{M^{j_1+j_2+\ldots+j_n-1}} \tag{A}$$

Vejvoda's assumption:

$$|f_i| \leqq L_B, \quad \left|\frac{\partial^{l+j_1+j_2+\ldots+j_n} f_i}{\partial x^l \partial y_1^{j_1} \ldots \partial y_n^{j_n}}\right| \leqq L_B^{l+1} \tag{B}$$

The assumption (B) is a special case of the assumption (A) if $L_A = M = L_B$.

Knichal's assumption supposes that the considered quantities fulfil the conditions in the $(r+1)$-th interval:

$$[f_i]_r = 0 \quad |f_i| \leqq 1$$

$$\left|\frac{\partial^{l+j_1+j_2+\ldots+j_n} f_i}{\partial x^l \partial y^{j_1} \ldots \delta y_n^{j_n}}\right| \leqq L_C^{l+j_1+j_2+\ldots+j_n} \tag{C}$$

The estimation under the assumption (A)

In the following computations we shall write L insted of L_A. Let the assumption (A) be fulfilled, then the following estimations can be done:

$$|D_m f_i| \leqq L^m M (1+n)^m$$

$$\left|\sum_{j=1}^{n} D_m\left(\frac{\partial f_i}{\partial y_j}\right)\right| \leqq L^{m+1}(1+n)^m n$$

$$\left|\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} D_m\left(\frac{d^2 f_i}{\partial y_{j_1} \partial y_{j_2}}\right)\right| \leqq \frac{L^{m+2}}{M}(1+n)^m n^2$$

$$\left|\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j_3=1}^{n} D_m\left(\frac{\partial^3 f_i}{\partial y_{j_1} \partial y_{j_2} \partial y_{j_3}}\right)\right| \leqq \frac{L^{m+3}}{M^2}(1+n)^m n^3 \tag{II.2}$$

$$\left|\frac{\mathrm{d}y_i}{\mathrm{d}x}\right| = |f_i| \leqq M,$$

$$\left|\frac{\mathrm{d}^2 y_i}{\mathrm{d}x^2}\right| \leqq LM(1+n)$$

$$\left|\frac{\mathrm{d}^3 y_i}{\mathrm{d}x^3}\right| = \leqq L^2 M(1+n)(1+2n);$$

$$\left|\frac{\mathrm{d}^4 y_i}{\mathrm{d}x^4}\right| \leqq L^3 M(1+n)(1+6n+6n^2)$$

$$\left|\frac{\mathrm{d}^5 y_i}{\mathrm{d}x^5}\right| \leqq L^4 M(1+n)(1+14n+36n^2+24n^3) \tag{II.3}$$

$$\left|\frac{\mathrm{d}^6 y_i}{\mathrm{d}x^6}\right| \leqq L^5 M(1+n)(1+30n+150n^2+240n^3+120n^4)$$

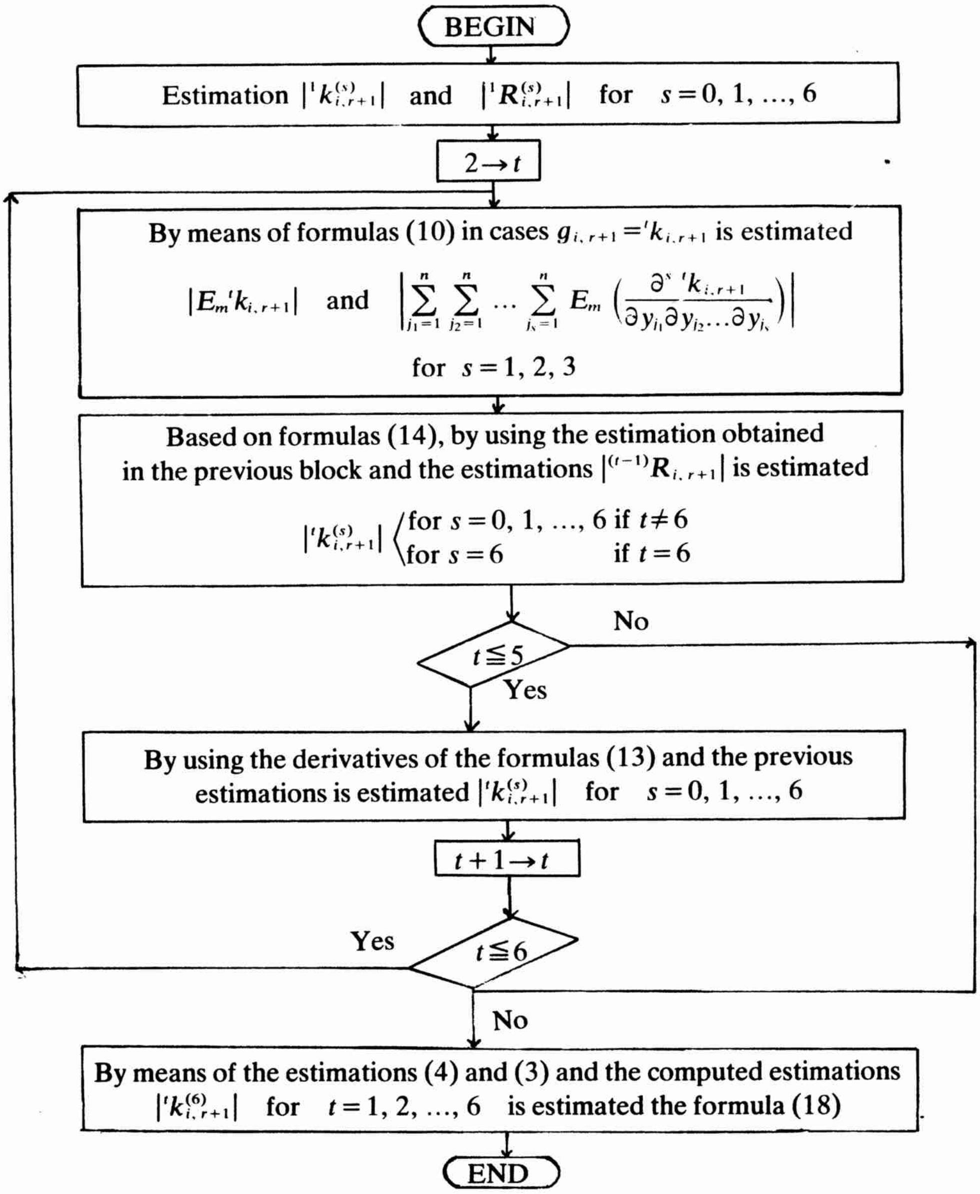
$$\left|\frac{\mathrm{d}^7 y_i}{\mathrm{d}x^7}\right| \leqq L^6 M(1+n)(1+62n+540n^2+1560n^3+1800n^4+720n^5) =$$

$$L^6M(1+63n+602n^2+2100n^3+3360n^4+2520n^5+720n^6)$$

$$|A_i| \leqq \frac{L^5M}{11\,520}\, n(1+n)(2+17n+41n^2+28n^3) =$$

$$= \frac{L^5M}{11\,520}(2n+19n^2+58n^3+69n^4+28n^5) \qquad \text{(II.4)}$$

The global flow chart for the estimation of formula (18)



The detailed description of the estimation of formula (18)

For the estimation of $\omega_{n,r+1}$ it is necessary to estimate $|A_i|$ and $|B_i|$. For the estimation of $|B_i|$ it is necessary to perform the estimations $|^t k_{i,r+1}^{(s)}|$ too. The computations of the estimations $|^t k_{i,r+1}^{(s)}|$ for $t = 1, 2, \dots, 6$; $s = 0, 1, \dots, 6$ are very complicated and from this reason the auxiliary expressions (*tKs*) and (*tRs*) are introduced. For the estimation of formula (18) the following auxiliary expressions are also introduced: (VR0), (VR1), (VR2), (VR3), (VR4), (HC1), (HC2), (HC3), (HC4). The computations of the auxiliary expressions were executed on the computer. In the following text an algorithm will be given for computing the auxiliary expressions and their evaluating for some given values of arguments n, hL. Computation on the computer was executed in fractional arithmetic.

The estimations of $|^t k_{i,r+1}^{(s)}|$ and $|^t R_{i,r+1}^{(s)}|$ can be expressed by means of (*tKs*) and (*tRs*) as follows:

$$|^1 k_{i,r+1}| \leqq M = M(1K0); \quad |^1 k'_{i,r+1}| = \dots = |^1 k_{i,r+1}^{(VI)}| = 0$$

$$(1K1) = (1K2) = \dots = (1K6) = 0; \quad |^1 R_{i,r+1}| \leqq hM = hM(1R0),$$

$$|^1 R'_{i,r+1}| \leqq M = M(1R1), \quad |^1 R''_{i,r+1}| = \dots = |^1 R_{i,r+1}^{(VI)}| = 0$$

$$(1R2) = (1R3) = \dots = (1R6) = 0$$

$$|^2 k_{i,r+1}^{(s)}| \leqq \frac{1}{6^s} L^s M(2Ks) \quad \text{for} \quad s = 0, 1, \dots, 6$$

$$|^2 R_{i,r+1}| \leqq hM = hM(2R0), \quad |^2 R'_{i,r+1}| \leqq M(2R1)$$

$$|^2 R_{i,r+1}^{(s)}| \leqq \frac{1}{6^{s-2}.8} L^{s-1} M(2Rs) \quad \text{for} \quad s = 2, 3, \dots, 6$$

$$|^3 k_{i,r+1}^{(s)}| \leqq \frac{1}{4^s} L^s M(3Ks) \quad \text{for} \quad s = 0, 1, \dots, 6$$

$$|^3 R_{i,r+1}| \leqq hM(3R0), \quad |^3 R'_{i,r+1}| \leqq M(3R1)$$

$$|^3 R_{i,r+1}^{(s)}| \leqq \frac{1}{2(4^{s-2})} L^{s-1} M(3Rs) \quad \text{for} \quad s = 2, 3, \dots, 6$$

$$|^4 k_{i,r+1}^{(s)}| \leqq \frac{1}{2^s} L^s M(4Ks) \quad \text{for} \quad s = 0, 1, \dots, 6$$

$$|^4 R_{i,r+1}| \leqq hM(4R0), \quad |^4 R'_{i,r+1}| \leqq M(4R1)$$

$$|^4 R_{i,r+1}^{(s)}| \leqq \frac{3}{4^{s-1}} L^{s-1} M(4Rs) \quad \text{for} \quad s = 2, 3, \dots, 6$$

$$|^5 k_{i,r+1}^{(s)}| \leqq \left(\frac{3}{4}\right)^s L^s M(5Ks) \quad \text{for} \quad s = 0, 1, \dots, 6$$

$$|^5R_{i,r+1}| \leqq hM(5R0), \quad |^5R'_{i,r+1}| \leqq M(5R1)$$

$$|^5R^{(s)}_{i,r+1}| \leqq \frac{1}{4^{s-2}} \frac{1}{14} L^{s-1} M(5Rs) \quad \text{for} \quad s = 2, 3, \ldots, 6$$

The estimations of $|^6k_{i,r+1}|$, $|^6k'_{i,r+1}|$, ..., $|^6k^{(V)}_{i,r+1}|$ we don't need.

$$|^6k^{(VI)}_{i,r+1}| \leqq \frac{L^6M}{7^4.2} (6K6)$$

For the estimation of formula (18) we get

$$\omega_{n,r+1} \leqq C_n = \frac{(hL)^5 hM}{3\,628\,800} [630n^2 + 5985n^3 + 18\,270n^4 + $$

$$+ 21\,735n^5 + 8820n^6 + (VRO)] \qquad \text{(II.5)}$$

Especially for $n = 1$ we get

$$\omega_{1,r+1} \leqq C_1 = \frac{(hL)^5 hM}{3\,628\,800} [50\,400 + (VR1)] = hM(HC1) \qquad \text{(II.6)}$$

At strict specialization C_n for $n = 1$ we should get 55 440 instead of 50 400. The improvement comes as a result of the summation of (and hence the better estimation) some terms within A_i, in case $n = 1$.

$$\omega_{2,r+1} \leqq C_2 = \frac{(hL)^5 hM}{3\,628\,800} [1\,602\,720 + (VR2)] = hM(HC2) \qquad \text{(II.7)}$$

$$\omega_{3,r+1} \leqq C_3 = \frac{(hL)^5 hM}{3\,628\,800} [13\,358\,520 + (VR3)] = hM(HC3) \qquad \text{(II.8)}$$

$$\omega_{4,r+1} \leqq C_4 = \frac{(hL)^5 hM}{3\,628\,800} [63\,453\,600 + (VR4)] = hM(HC4) \qquad \text{(II.9)}$$

The algorithm of the computation of
(*tKs*), (*tRs*), (*HC*1), (*HC*2), (*HC*3), (*HC*4), (*VR*0), (*VR*1), (*VR*2), (*VR*3), (*VR*4)

I. Read and store (save)

1) $(1K0) = 1$, $(1K1) = (1K2) = \ldots = (1K6) = 0$
2) $(2Ks) = (1 + n)^s$ for $s = 0, 1, \ldots, 6$

II. 1) Read $(2Rs)$ and hold all the $(2Rs)$ for $s = 0, 1, \ldots, 6$ until all the $(3Ks)$ are computed.

$(2R0) = 1$,

$$(2R1) = 1 + \frac{1}{8}hL + \frac{1}{8}nhL,$$

$$(2R2) = 2 + 2n + \frac{1}{6}hL + \frac{1}{3}nhL + \frac{1}{6}n^2hL,$$

$$(2R3) = 3 + 6n + 3n^2 + \frac{1}{6}hL + \frac{1}{2}nhL + \frac{1}{2}n^2hL + \frac{1}{6}n^3hL,$$

$$(2R4) = 4 + 12n + 12n^2 + 4n^3 + \frac{1}{6}hL + \frac{2}{3}nhL + n^2hL + \frac{2}{3}n^3hL + \frac{1}{6}n^4hL,$$

$$(2R5) = 5 + 20n + 30n^2 + 20n^3 + 5n^4 + \frac{1}{6}hL + \frac{5}{6}nhL + \frac{5}{3}n^2hL + \frac{5}{3}n^3hL + \\ + \frac{5}{6}n^4hL + \frac{1}{6}n^5hL$$

$$(2R6) = 6 + 30n + 60n^2 + 60n^3 + 30n^4 + 6n^5 + \frac{1}{6}hL + nhL + \frac{5}{2}n^2hL + \\ + \frac{10}{3}n^3hL + \frac{5}{2}n^4hL + n^5hL + \frac{1}{6}n^6hL$$

2) Based on the given $(2Rs)$ and further data $(3Ks)$ will be computed for $s = 0, 1, \ldots, 6$ as follows:

a) Store and save the following expression

$$(Z3) = 1 + n + \frac{1}{8}nhL + \frac{1}{8}n^2hL$$

b) Compute, store and write

$$(3K0) = 1,$$

$$(3K1) = (Z3),$$

$$(3K2) = (Z3)^2 + \frac{1}{2}n(2R2)$$

$$(3K3) = (Z3)^3 + \frac{3}{2}n(Z3)(2R2) + \frac{1}{3}n(2R3)$$

$$(3K4) = (Z3)^4 + 3n(Z3)^2(2R2) + \frac{4}{3}n(Z3)(2R3) + \frac{2}{9}n(2R4) + \\ + \frac{3}{4}n^2(2R2)^2,$$

$$(3K5) = (Z3)^5 + 5n(Z3)^3(2R2) + \frac{10}{3}n(Z3)^2(2R3) + \frac{10}{9}n(Z3)(2R4) + \\ + \frac{4}{27}n(2R5) + \frac{15}{4}n^2(Z3)(2R2)^2 + \frac{5}{3}n^2(2R2)(2R3)$$

$$(3K6) = (Z3)^6 + \frac{15}{2}n(Z3)^4(2R2) + \frac{20}{3}n(Z3)^3(2R3) + \frac{10}{3}n(Z3)^2(2R4) +$$

$$+\frac{8}{9}n(Z3)(2R5)+\frac{8}{81}n(2R6)+\frac{45}{4}n^2(Z3)^2(2R2)^2+10n^2(Z3)\times(2R2)(2R3)+$$

$$+\frac{5}{3}n^2(2R2)(2R4)+\frac{10}{9}n^2(2R3)^2+\frac{15}{8}n^3(2R2)^3$$

In what follows we need not save $(2Rs)$. All $(3Ks)$ are to be saved and written.

III. 1) Compute the following expressions $(3Rs)$ for $s=0, 1, \ldots, 6$ hold all results $(3Rs)$ until all $(4Ks)$ will be computed.

$$(3R0)=4,$$

$$(3R1)=4+\frac{1}{4}hL(2K1)+\frac{1}{2}hL(3K1)$$

$$(3R2)=(2K1)+2(3K1)+\frac{1}{12}hL(2K2)+\frac{1}{4}hL(3K2)$$

$$(3R3)=(2K2)+3(3K2)+\frac{1}{18}hL(2K3)+\frac{1}{4}hL(3K3)$$

$$(3R4)=\frac{8}{9}(2K3)+4(3K3)+\frac{1}{27}hL(2K4)+\frac{1}{4}hL(3K4)$$

$$(3R5)=\frac{20}{27}(2K4)+5(3K4)+\frac{2}{81}hL(2K5)+\frac{1}{4}hL(3K5)$$

$$(3R6)=\frac{16}{27}(2K5)+6(3K5)+\frac{4}{243}hL(2K6)+\frac{1}{4}hL(3K6)$$

2) Based on (3Rs) and further data (4Ks) will be computed and saved for $s=0, 1, \ldots, 6$ *as follows*:

a) Store and hold the following expression

$$(Z4)=1+4n+\frac{3}{4}nhL+\frac{3}{4}n^2hL+\frac{1}{16}n^2(hL)^2+\frac{1}{16}n^3(hL)^2$$

until all $(4Ks)$ are be computed.

b) Compute, store and write the results of the following expressions:

$$(4K0)=1,$$

$$(4K1)=(Z4),$$

$$(4K2)=(Z4)^2+n(3R2)$$

$$(4K3)=(Z4)^3+3n(Z4)(3R2)+\frac{1}{2}n(3R3),$$

$$(4K4)=(Z4)^4+6n(Z4)^2(3R2)+2n(Z4)(3R3)+\frac{1}{4}n(3R4)+3n^2(3R2)^2$$

$$(4K5)=(Z4)^5+10n(Z4)^3(3R2)+5n(Z4)^2(3R3)+\frac{5}{4}n(Z4)(3R4)+$$

$$+\frac{1}{8}n(3R5)+15n^2(Z4)(3R2)^2+5n^2(3R2)(3R3)$$

$$(4K6)=(Z4)^6+15n(Z4)^4(3R2)+10n(Z4)^3(3R3)+\frac{15}{4}n(Z4)^2(3R4)+$$
$$+\frac{3}{4}n(Z4)(3R5)+\frac{1}{16}n(3R6)+45n^2(Z4)^2(3R2)^2+$$
$$+30n^2(Z4)(3R2)(3R3)+\frac{15}{4}n^2(3R2)(3R4)+\frac{5}{2}n^2(3R3)^2+$$
$$+15n^3(3R2)^2$$

IV. 1) Compute the results of the following $(4Rs)$ for $s=0, 1, \ldots, 6$. Save all $(4Rs)$ until all $(5Ks)$ are be computed.

$$(4R0)=1,$$
$$(4R1)=1+\frac{3}{8}hL(4K1),$$
$$(4R2)=(4K1)+\frac{1}{4}hL(4K2)$$
$$(4R3)=3(4K2)+\frac{1}{2}hL(4K3),$$
$$(4R4)=8(4K3)+hL(4K4),$$
$$(4R5)=20(4K4)+2hL(4K5),$$
$$(4R6)=48(4K5)+4hL(4K6)$$

2) Based on $(4Rs)$ and following data $(5Ks)$ will be computed for $s=0, 1, \ldots, 6$.

a) Store and save the following expression

$$(Z5)=1+n+\frac{3}{8}nhL+\frac{3}{2}n^2hL+\frac{9}{32}n^2(hL)^2+\frac{9}{32}n^3(hL)^2+$$
$$+\frac{3}{128}n^3(hL)^3+\frac{3}{128}n^4(hL)^3$$

b)

$$(5K0)=1,$$
$$(5K1)=(Z5),$$
$$(5K2)=(Z5)^2+n(4R2)$$
$$(5K3)=(Z5)^3+3n(Z5)(4R2)+\frac{1}{3}n(4R3),$$
$$(5K4)=(Z5)^4+6n(Z5)^2(4R2)+\frac{4}{3}n(Z5)(4R3)+\frac{1}{9}n(4R4)+3n^2(4R2)^2$$
$$(5K5)=(Z5)^5+10n(Z5)^3(4R2)+\frac{10}{3}n(Z5)^2(4R3)+\frac{5}{9}n(Z5)(4R4)+$$

$$+\frac{1}{27}n(4R5)+15\,n^2(Z5)(4R2)^2+\frac{10}{3}n^2(4R2)(4R3)$$

$$(5K6)=(Z5)^6+15n(Z5)^4(4R2)+\frac{20}{3}n(Z5)^3(4R3)+\frac{5}{3}n(Z5)^2(4R4)+$$
$$+\frac{2}{9}n(Z5)(4R5)+\frac{1}{81}n(4R6)+45n^2(Z5)^2(4R2)^2+$$
$$+20n^2(Z5)(4R2)(4R3)+\frac{5}{3}n^2(4R2)(4R4)+\frac{10}{9}n^2(4R3)^2$$
$$+15n^3(4R2)^3$$

V. 1) Compute and write the results of the following expressions $(5Rs)$ for $s=0, 1, \ldots, 6$. Save all $(5Rs)$ until $(6Ks)$ is be computed

$$(5R0)=\frac{39}{7},$$

$$(5R1)=\frac{39}{7}+\frac{1}{14}hL(2K1)+\frac{3}{7}hL(3K1)+\frac{6}{7}hL(4K1)+\frac{6}{7}hL(5K1),$$

$$(5R2)=2(2K1)+12(3K1)+24(4K1)+24(5K1)+\frac{1}{6}hL(2K2)+$$
$$+\frac{3}{2}hL(3K2)+6hL(4K2)+9hL(5K2),$$

$$(5R3)=2(2K2)+18(3K2)+72(4K2)+108(5K2)+\frac{1}{9}hL(2K3)+$$
$$+\frac{3}{2}hL(3K3)+12hL(4K3)+27hL(5K3),$$

$$(5R4)=\frac{16}{9}(2K3)+24(3K3)+192(4K3)+432(5K3)+\frac{2}{27}hL(2K4)+$$
$$+\frac{3}{2}hL(3K4)+24hL(4K4)+81hL(5K4)$$

$$(5R5)=\frac{40}{27}(2K4)+30(3K4)+480(4K4)+1620(5K4)+\frac{4}{81}hL(2K5)+$$
$$+\frac{3}{2}hL(3K5)+48hL(4K5)+243hL(5K5)$$

$$(5R6)=\frac{32}{27}(2K5)+36(3K5)+1152(4K5)+5832(5K5)+$$
$$+\frac{8}{243}hL(2K6)+\frac{3}{2}hL(3K6)+96hL(4K6)+729hL(5K6)$$

2) Based on $(5Rs)$ and the following data, compute $(6K6)$.

a) Store and save the following expression:

$$(Z6)=7+39n+\frac{31}{2}nhL+\frac{67}{2}n^2hL+\frac{57}{8}n^2(hL)^2+\frac{111}{8}n^3(hL)^2+$$

$$+\frac{33}{16}n^3(hL)^3+\frac{33}{16}n^4(hL)^3+\frac{9}{64}n^4(hL)^4+\frac{9}{64}n^5(hL)^4$$

b) $$(6K6)=\frac{2}{49}(Z6)^6+\frac{15}{7}n(Z6)^4(5R2)+5n(Z6)^3(5R3)+$$
$$+\frac{105}{16}n(Z6)^2(5R4)+\frac{147}{32}n(Z6)(5R5)+\frac{343}{256}n(5R6)+$$
$$+\frac{45}{2}n^2(Z6)^2(5R2)^2+\frac{105}{2}n^2(Z6)(5R2)(5R3)+$$
$$+\frac{735}{32}n^2(5R2)(5R4)+\frac{245}{16}n^2(5R3)^2+\frac{105}{4}n^3(5R2)^3$$

VI. Compute, store and write

1) $$(VR0)=720nhL+45\,360n^2hL+433\,440n^3hL+1\,512\,000n^4hL+$$
$$+2\,419\,200n^5hL+1\,814\,400n^6hL+518\,400n^7hL+$$
$$+\frac{7}{16}nhL(3K6)+\frac{21}{2}nhL(4K6)+\frac{5103}{16}nhL(5K6)+$$
$$+\frac{4}{49}nhL(6K6) \qquad \text{(II.10)}$$

2) Compute and write

$$(VRj)=(VR0)_{n=j} \quad \text{for} \quad j=1,2,3,4 \qquad \text{(II.11)}$$

Remark: (VRJ) is a function of hL and we obtain it when whithin (VRO) we put $n=j$.

Compute the values and write

$$(HC1)=\frac{(hL)^5}{3\,628\,800}[50\,400+(VR1)] \qquad \text{(II.12)}$$

for $hL=0{,}01;\ 0{,}02;\ \ldots;\ 0{,}20$

$$(HC2)=\frac{(hL)^5}{3\,628\,800}[1\,602\,720+(VR2)] \qquad \text{(II.13)}$$

for $hL=0{,}01;\ 0{,}02;\ \ldots;\ 0{,}20$

For the same values of the hL as they were computed for $(HC1)$ and $(HC2)$, compute and write the $(HC3)$ and $(HC4)$.

$$(HC3)=\frac{(hL)^5}{3\,628\,800}[13\,358\,520+(VR3)] \qquad \text{(II.14)}$$

$$(HC4)=\frac{(hL)^5}{3\,628\,800}[63\,453\,600+(VR4)] \qquad \text{(II.15)}$$

$(VR0)$ is the high degree polynomial of the variables n, h, L. The number of the members of the polynomial in this case is 175 [5]. $(VR1)$, $(VR2)$, $(VR3)$ and $(VR4)$ are the polynomials of the 25th degree of the variables (hL). These polynomials can be written as follows

$$(VRj)=\sum_{i=1}^{25}\frac{a_{ji}}{b_{ji}}(hL)^i \quad \text{for} \quad j=1,2,3,4 \quad \text{(see [5])}$$

The results $(HC1)$, $(HC2)$, $(HC3)$ and $(HC4)$ obtained by the computer are stated in the following page.

The assumption (B) is a special case of the assumption (A).

For the estimation of formula (18) under the assumption (C) we get:

$$\omega_{n,r+1}\leqq C_n=$$
$$=\frac{L^5h^6}{3\,628\,800}[630n^2+3465n^3+4095n^4+630n^5+(VR0)] \qquad \text{(II.16)}$$

In the case $n=1$

$$\omega_{1,r+1}\leqq C_1=\frac{h(hL)^5}{3\,628\,800}[7560+(VR1)]=h(HC1) \qquad \text{(II.17)}$$

At the strict specialization C_n for $n=1$ we should get the first member in the brackets 8820 instead of 7560. The improvement comes as a result of the summation and hence the better estimation of some members within A_i for $n=1$.

$$\omega_{2,r+1}\leqq C_2=\frac{h(hL)^5}{3\,628\,800}[115\,920+(VR2)]=h(HC2) \qquad \text{(II.18)}$$

$$\omega_{3,r+1}\leqq C_3=\frac{h(hL)^5}{3\,628\,800}[584\,010+(VR3)]=h(HC3) \qquad \text{(II.19)}$$

$$\omega_{4,r+1}\leqq C_4=\frac{h(hL)^5}{3\,628\,800}[1\,925\,280+(VR4)]=h(HC4) \qquad \text{(II.20)}$$

The constant L in this case in a constant $L=L_C$ under the assumption (C). The (VRO) is a high degree polynomial of the variables n, h, L. The number of the members of the polynomial in this case is 154 [5]. $(VR1)$, $(VR2)$, $(VR3)$ and $(VR4)$ are the polynomials of the variables (hL) of the 25th degree. These polynomials can be written as follows:

$$(VRj)=\sum_{i=1}^{25}\frac{\bar{a}_{ji}}{\bar{b}_{ji}}(hL)^i \quad \text{for} \quad j=1,2,3,4 \ [5]$$

The results $(HC1)$, $(HC2)$, $(HC3)$ and $(HC4)$ obtained by computer are stated in the following page.

Let us compare the formula (II.6) with the error estimation from the point of

The computed values of (*HC*1), (*HC*2), *HC*3), (*HC*4) in case assumption (A)

HL	(*HC*1)	(*HC*2)	(*HC*3)	(*HC*4)
0.01	0.238407361499788*D*−09	0.190841033833644*D*−07	0.283107756340062*D*−06	0.201457090855117*D*−05
0.02	0.160496463342388D−07	0.134676250788209D−05	0.209050461185422D−04	0.155569125012877D−03
0.03	0.192682531614079D−06	0.169261106381450D−04	0.274754441500224D−03	0.213725309346159D−02
0.04	0.114156335875152D−05	0.104932214029822D−03	0.178067229638288D−02	0.144736656758410D−01
0.05	0.459247162453476D−05	0.441602309203847D−03	0.783205578419367D−02	0.664975351164625D−01
0.06	0.144622943240502D−04	0.145447249695303D−02	0.269530075576448D−01	0.238960089641740D 00
0.07	0.384606961205483D−04	0.404470416381049D−02	0.782953109653511D−01	0.724593905410508D 00
0.08	0.903765510613314D−04	0.993687044225557D−02	0.200879911693360D 00	0.193993656842047D 01
0.09	0.193214002374205D−03	0.222065982255139D−01	0.468702358592429D 00	0.472160880926179D 01
0.10	0.383380057477001D−03	0.460520830547862D−01	0.101457124739847D 01	0.106577510832865D 02
0.11	0.716154641359356D−03	0.898939137265390D−01	0.206666946942279D 01	0.226303359941893D 02
0.12	0.127272133148176D−02	0.166912264120523D 00	0.400336405816550D 01	0.456801392838208D 02
0.13	0.216907357678286D−02	0.297158759196444D 00	0.743377557749669D 01	0.883564453633279D 02
0.14	0.356715300858321D−02	0.510414642412608D 00	0.133142087568749D 02	0.164783409822101D 03
0.15	0.568867462859000D−02	0.850016510556526D 00	0.231141282168393D 02	0.297773992680942D 03
0.16	0.883208985591903D−02	0.137791429185659D 01	0.390495672648434D 02	0.523452683556063D 03
0.17	0.133932852812959D−01	0.218130233642281D 01	0.644079075331026D 02	0.898033301069529D 03
0.18	0.198905974105915D−01	0.338122349027051D 01	0.103994734466257D 03	0.150763443947261D 04
0.19	0.289950283531811D−01	0.514368242166859D 01	0.164744242544884D 03	0.248236364979141D 04
0.20	0.415660205656582D−01	0.769379259420655D 01	0.256542295274877D 03	0.401627065741724D 04

The computed values of (*HC*1), (*HC*2), (*HC*3), (*HC*4) in case assumption

HL	(*HC*1)	(*HC*2)	(*HC*3)	(*HC*4)
0.01	0.146890883008643D−09	0.109961066945617D−07	0.158194677830390D−06	0.110450697487045D−05
0.02	0.988208684210749D−08	0.771072684032567D−06	0.115487102130843D−04	0.839129907710428D−04
0.03	0.118373289666972D−06	0.962196092567156D−05	0.149985174589928D−03	0.113371727769990D−02
0.04	0.699464088284283D−06	0.592143127810270D−04	0.960376381726215D−03	0.754929978756212D−02
0.05	0.280603798454230D−05	0.247354555670727D−03	0.417303978440199D−02	0.341016545166036D−01
0.06	0.881105226695214D−05	0.808614417956640D−03	0.141866895048933D−01	0.120478757243881D 00
0.07	0.233629661661945D−04	0.223179763690547D−02	0.407090342264785D−01	0.359149057375957D 00
0.08	0.547357657150031D−04	0.544175347274431D−02	0.103171743599963D 00	0.945249494978205D 00
0.09	0.116666812596622D−03	0.120693634442482D−01	0.237783306264235D 00	0.226159532088095D 01
0.10	0.230793187653930D−03	0.248403968824386D−01	0.508416332365520D 00	0.501817678868360D 01
0.11	0.429811740078934D−03	0.481217847116969D−01	0.102295154863552D 01	0.104741406078320D 02
0.12	0.761513042824966D−03	0.886743984940954D−01	0.195727525503979D 01	0.207823665512090D 02
0.13	0.129385734998393D−02	0.156673081240993D 00	0.358984595840607D 01	0.395130507802982D 02
0.14	0.212128113194496D−02	0.267068616569307D 00	0.635066168764299D 01	0.724346447485867D 02
0.15	0.337247584154709D−02	0.441387114493648D 00	0.108897241596353D 02	0.128661013114762D 03
0.16	0.521987226970277D−02	0.710078356764037D 00	0.181714892412660D 02	0.222311920642668D 03
0.17	0.789114360816966D−02	0.111555605110951D 01	0.296038496907269D 02	0.374887923307221D 03
0.18	0.116830231294257D−01	0.171609362985576D 01	0.472123224788389D 02	0.618625866177953D 03
0.19	0.169779003406080D−01	0.259079629774851D 01	0.738737274041781D 02	0.100119897744105D 04
0.20	0.242633442895212D−01	0.384585080999190D 01	0.113625480036354D 03	0.159221470131033D 04

view of the error for one differential equation

$$T_m = \gamma_m h^{m+1} + 0(h^{m+2}) \quad [6]$$

In our case $m = 5$. For $n = 1$ we can leave the term which contains $(VR1)$ and to consider it as a quantity of the order $0(h^7)$. Then we get

$$\gamma_5 \leqq \frac{50\,400}{3\,628\,800} L^5 M = \frac{1}{72} L^5 M \doteq 0{,}0139 ML^5 \tag{II.21}$$

Let us compare this result with the estimation of the formula $RKN5$ [7].

$$\gamma_5 \leqq \frac{91\,397}{194\,400} ML^5 \doteq 0{,}4702 ML^5 \quad [8]$$

Thus, by the comparison between the estimations γ_5 for $RKH5$ and γ_5 for $RKN5$, we can find out that the estimation γ_5 for $RKN5$ is approximately 33,8507 times greater than the estimation γ_5 for $RKH5$, without consideration of the values of the constants M, L.

Let us try to estimate the theoretical formula (I.18) for the error of the formula $RKH6$. If the assumption (A) is fulfilled then for estimation $|A_i|$ there holds:

$$|A_i| \leqq \frac{L^6 M}{37\,800\,000} (11 + 922n + 11\,990n^2 + 54\,390n^3 + 108\,555n^4 + \\ + 98\,556n^5 + 33\,312n^6)$$

For the estimation of formula (I.18) we get

$$\omega_{n,r+1} \leqq C_n = \frac{(hL)^6 hM}{37\,800\,000} [11n + 922n^2 + 11\,990n^3 + 54\,390n^4 + \\ + 108\,555n^5 + 98\,556n^6 + 33\,312n^7 + (VR0)] \tag{II.22}$$

where (VRO) should be obtained analogically as in the case of $RKH5$. The computations necessary for the expression of $(VR0)$ are so complicated and their range is so large that up to now it was not possible to perform it even by the use of a computer. The $(VR0)$ is the high degree polynomial of the variables n, h, L.

In case for $n = 1$, some terms within (I.18) can be summed up and estimated in a better way.

$$\omega_{1,r+1} \leqq C_1 = \frac{(hL)^6 hM}{37\,800\,000} [229\,096 + (VR1)] \tag{II.23}$$

The $(VR1)$ means $(VR0)_{n=1}$ and it is the polynomial which can be written as follows:

$$\sum_i \frac{a_{1i}}{b_{1i}} (hL)^i$$

Let us compare the formula (II.23) with the estimation from the point of view of the formula of the error for one differential equation

$$T_m = \gamma_m h^{m+1} + 0(h^{m+2}) \ [6]$$

In this case $m = 6$. For $n = 1$ we can leave the term which contains $(VR1)$ and to consider it as a quantity of the order $0(h^8)$. Then we get

$$\gamma_6 \leqq \frac{229\ 096}{37\ 800\ 000} ML^6 \doteq 0{,}0061 ML^6 \tag{II.24}$$

The result (II.24) agrees with the result stated in [9].

Analogically for the estimation (I.18) at the assumption (C) we should get

$$\omega_{n,\,r+1} \leqq C_n^* = \frac{L^6 h^7}{37\ 800\ 000} [11n + 856n^2 + 7545n^3 + 15\ 430n^4 + \\ + 8270n^5 + 1200n^6 + (VR0)] \tag{II.25}$$

The computation of the $(VR0)$ was not successful even in this case.

$$\omega_{1,\,r+1} \leqq C_1^* = \frac{h^7 L^6}{37\ 800\ 000} [24\ 052 + (VR1)] \tag{II.26}$$

In the case of the comparation with the formula $T_m^* = \gamma_m^* h^{m+1} + 0(h^{m+2})$ we get

$$\gamma_6^* \leqq \frac{24\ 052}{37\ 800\ 000} L^6 \doteq 0{,}000636 L^6.$$

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Anna Valková
Katedra numerickej matematiky PFUK,
Matematický pavilón
Mlynská dolina
842 15 Bratislava

SÚHRN

ODHAD LOKÁLNEJ CHYBY VZORCOV TYPU RUNGE—KUTTA—HUŤA PIATEHO A ŠIESTEHO RÁDU PRE SYSTÉM DIFERENCIÁLNYCH ROVNÍC

Anna Valková, Bratislava

V tomto článku sú uvedené odhady lokálnej chyby približného riešenia začiatočnej úlohy pre systém diferenciálnych rovníc prvého rádu v prípade jeho riešenia pomocou vzorcov typu Runge—Kutta—Huťa 5. a 6. rádu.

Môj prístup je analogický prístupu, ktorý použil Vejvoda pre odhad chyby klasických vzorcov typu Runge—kutta 4. rádu. [2]

Porovnaním odhadov vzorcov typu Runge—Kutta—Huťa 5. rádu a vzorcov typu Runge—Kutta—Nyström 5. rádu pre jednu diferenciálnu rovnicu sa zistilo, že odhad pre RKN5 [8] je približne 33,85-krát horší ako získaný odhad pre RKH5, bez ohľadu na to aké sú konštanty M, L.

РЕЗЮМЕ

ОЦЕНКА ОШИБКИ ФОРМУЛ ТИПА РУНГЕ—КУТТА—ХУТЬЯ ПЯТОГО И ШЕСТОГО ПОРЯДКА

Анна Валкова, Братислава

В этой статьи предлагается оценка ошибки формул Рунге—Кутта—Хутья пятого и шестого порядка [1] для численного решения системы n дифференцияльных уравнений первого порядка. Даются верхние границы ишибки формулы Хутья для одного шага.

Показывается что оценка для формулы Хутья пятого порядка лучшая чем оценка для формулы Нюстрема тоже пятого порядка [8].