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THE NP-COMPLETENESS OF THE HAMILTONIAN CYCLE PROBLEM IN BIPARTITE CUBIC PLANAR GRAPHS

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It is well known that the hamiltonian cycle problem is NP-complete even for cubic planar graphs [2], or cubic planar digraphs [5], or bipartite graphs [4]. The purpose of this paper is to note that the first two results can be extended to bipartite graphs (digraphs). The same is true for the hamiltonian path problem. To prove these results it is sufficient to change only some couplings in the constructions from [2] and [5]. Therefore we shall not give full constructions, but the reader will be referred to these papers (in fact, the paper [5] will suffice).

Our terminology is based on [3]. If u is a point of a digraph then the sum of the indegree of u and the outdegree of u is called simply the degree of u. A graph or digraph is cubic if each point has degree 3. Let F be a face of a plane block (a graph without cutpoints), then the length (the number of lines) of the boundary of F is called the length of F.

Before stating the main results, we give the following assertion whose proof is obvious.

Lemma. If the length of each interior face of a plane block G is even then the graph G is bipartite.

Theorem 1. Both the hamiltonian cycle problem and the hamiltonian path problem are *NP*-complete even in the case of bipartite cubic planar graphs.

Proof. The proof differs from that of [5] only in some details: (1) Everywhere consider undirected lines instead of arcs (directed lines). (2) The "3-input or" realize as in *Fig. 1* and the "exclusive-or line" as in *Fig. 2.* (3) Replace the pairs of vertices of degree 2 by *Fig. 3.* (4) The crossing of exclusive-or lines is solved as in *Fig. 5* of [5], but now all the lines are undirected.

It is easy but tedious to verify that the obtained graph has the required properties and therefore the details are left to the reader. We recommend to use the foregoing lemma.

Theorem 2. Both the hamiltonian cycle problem and the hamiltonian path problem are *NP*-complete even in the case of bipartite cubic planar digraphs.

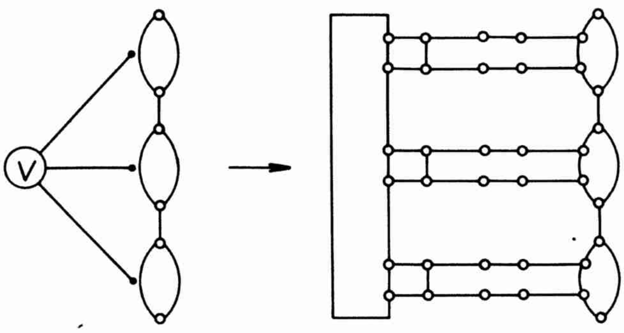

Fig. 1

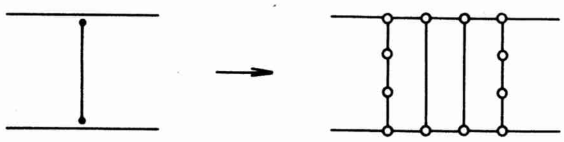

Fig. 2

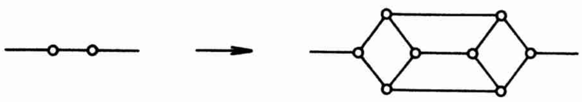

Fig. 3

To prove Theorem 2 it suffices to verify that the digraph constructed in [5] is bipartite. This is again left to the reader (use Lemma).

Remark. Garey, Johnson and Tarjan [2] have proved the NP-completeness also in the case of 3-connected cubic planar graphs. Therefore one might want to strengthen Theorem 1 by adding the 3-connectivity constraint. This task remains open. Note that the result of [2] is based on the Tutte graph [6] which is 3-connected, cubic, planar, and nonhamiltonian. While the Horton graph (see [1]) is an example of nonhamiltonian 3-connected cubic bipartite graph, no such example is known and Barnette (1970) conjectured that there is none (cf. [1]) if the planarity is required too.

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NP-ÚPLNOSŤ PROBLÉMU HAMILTONOVSKEJ KRUŽNICE V PÁRNYCH KUBICKÝCH PLANÁRNYCH GRAFOCH

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Ukazuje sa, že problém zistiť, či má graf hamiltonovskú kružnicu je *NP*-úplný dokonca pre párne kubické planárne grafy. Rovnaký výsledok platí aj pre problém hamiltonovskej cesty. Analogický výsledok je daný aj pre digrafy.

РЕЗЮМЕ

NP-ПОЛНОТА ПРОБЛЕМЫ ГАМИЛЬТОНОВА ЦИКЛА В ДВУДОЛЬНЫХ КУБИЧЕСКИХ ПЛОСКИХ ГРАФАХ

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Показано, что проблема нахождения гамильтонова цикла — *NP*-полная и для двудольных кубических плоских графов. Это утверждение имеет место и для проблемы гамильтоновой пути. Аналогичный результат дан и для ориентированных графов.

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ON A GENERALIZATION OF MENGER'S THEOREM

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1. Introduction

There are many results concerning the connectivity and the edge connectivity of a graph G. The clasical result in this direction is the well known Menger's theorem. For applications it is often useful to investigate the paths with bounded lengths. In [1] authors L. Lovász, V. Neumann-Lara and M. Plummer investigate the relation between the maximal number of disjoint paths with bounded length connecting two non adjacent vertices in a connected graph G and the minimal number of vertices in G the removing of which destroys all those paths.

In the present paper we study the relation between the maximal number of edge disjoint paths with the length less than n and the minimal number of edges removing of which interrupts all those paths.

2. Formulation of the problem and general results

We consider nonoriented finite graphs without loops or multiple edges in the sence of [2]. Let u and v be two disjoint vertices of a graph G. Let n be an arbitrary integer. We denote by $B_n(u, v)$ the maximal number of edge disjoint paths connecting the vertices u and w with the length not exceeding n. By $H_n(u, v)$ we denote the minimal number of the edges of the graph G removing of which implies destroying of all the $u-v$ paths with the length less or equal n.

From the Menger's theorem we obtain

$$B_n(u, v) = H_n(u, v)$$

in the case $n \geqq q$ where q is the number of edges of grah G.

It can be proved that in case $n < q$ there holds the inequality

$$B_n(u, v) \leqq H_n(u, v) \tag{1}$$

One can easy show that the inequality (1) may be sharp. For the graph G in

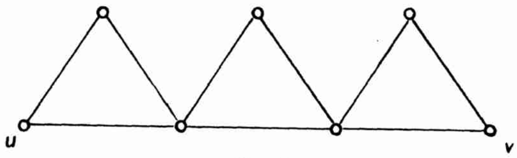


Fig. 1

the *Fig. 1* we have

$$B_4(u, v) = 1\, H_4(u, v) = 2 \tag{2}$$

In the following considerations we try to estimate the ratio

$$\frac{H_n(u, v)}{B_n(u, v)}$$

One can easy prove that

$$1 \leqq \frac{H_n(u, v)}{B_n(u, v)} \leqq n$$

This trivial estimation can be improved as follows from the following theorem.

Theorem 1. Let n be an integer and u, v be two disjoint vertices of a connected graph G. Denote $m = n - d(u, v)$ where $d(u, v)$ is a distance of u to v. Then

$$\frac{H_n(u, v)}{B_n(u, v)} \leqq m + 1$$

holds.

Proof. The proof proceeds by the induction on m. Denote $n_0 = d(u, v)$.

1° For $m = 0$ we have $n = d(u, v) = n_0$. That means we consider only the $u - v$ geodesics(shortest paths joining vertices u and b). We form a digraph D associated to G by the following manner. Graph G and digraph D have equal sets of vertices, and if xy is an edge of G then $\overline{(x, y)}$ is an arc in D if

$$d(x, v) > d(y, v)$$

Two statements follow from this construction:

(1) every $u - v$ geodesic yields a dipath from u to v,
(2) every $u - v$ dipath in D corresponds to $u - v$ geodesic in G.

The first statement is obvious. Suppose (2) is false, then the corresponding $u - v$ path has the length at least $n_0 + 1$. Let (u, x) be the first edge of such a path. In G $d(u, x) = 1$ holds and $d(x, v) = n_0 + 1 - d(u, x) = n_0$. That means $d(x, v) = d(u, v)$ and D does not contain the edge $\overline{(u, x)}$, what is a contradiction. From (1) and (2) we obtain

$$B_{n_0}(u, v) = B^D_{n_0}(u, v)$$
$$H_{n_0}(u, v) = H^D_{n_0}(u, v)$$

In the digraph D all $u - v$ dipaths have the length n_0 and from the Menger's theorem it follows that

$$B^D_{n_0}(u, v) = H^D_{n_0}(u, v)$$

and hence

$$B_{n_0}(u, v) = H_{n_0}(u, v)$$

what is the statement of the theorem for $m = 0$.

2° For $m > m_0 \geqq 0$ it holds $n > d(u, v) = n_0$ and obviously

$$B_{n_0}(u, v) = B_n(u, v).$$

Using the first step of the induction we have $B_{n_0}(u, v) = H_{n_0}(u, v)$. Let X denote a set of edges the removing of which destroys all $u - v$ paths with the length not exceeding n_0. Let V be the vertex set of G and H be a set of edges of G. Then $G - X$ denotes the graph $G - X = (V, H - X)$. For the number of edges of X we have

$$|X| = H_{n_0}(u, v) = B_{n_0}(u, v) \leqq B_n(u, v).$$

For the distance between u and v in $G - X$

$$d_{G-X}(u, v) > n_0$$

holds.

There are two possibilities

a) X covers all $u - v$ paths with the length $\leqq n$. Then

$$H_n(u, v) = H_{n_0}(u, v) = B_{n_0}(u, v) \leqq B_n u, v$$

and hence

$$\frac{H_n(u, v)}{B_n(u, v)} \leqq 1$$

b) X does not cover all $u - v$ paths with the length $\leqq n$. Denote $m_0 = n - d_{G-X}(u, v)$. As

$$d_{G-X}(u, v) > d(u, v) = n_0$$

then

$$m_0 < n - n_0 = m$$

By the induction hypothesis in the graph $G - X$

$$\frac{H_n^{G-X}(u, v)}{B_n^{G-X}(u, v)} \leqq m_0 + 1$$

holds.

Let Y be a set of edges of G destroing all $u - v$ paths with length $\leqq n$, then Y is a union of X and a set of all edges destroying $u - v$ paths with the length $\leqq n$ in the graph $G - X$.

$$|Y| = |X| + H_n^{G-X}(u, v)|X| + (m_0 + 1)B_n^{G-X}(u, v)$$
$$B_n(u, v) + (m_0 + 1)B_n(u, v)(m_0 + 2)B_n(u, v)$$
$$(m + 1)B_n(u, v).$$

Since $|Y| = H_n(u, v)$, we have

$$\frac{H_n(u, v)}{B_n(u, v)} \leqq m + 1$$

q. e. d.

Now we find some estimations of the ratio $\dfrac{H_n(u, v)}{B_n(u, v)}$ depending only on n.

Theorem 2. Let n be an arbitrary positive integer and u, v two distinct vertices of a connected graph G. Then

$$\frac{H_n(u, v)}{B_n(u, v)} \leqq \left[\frac{n + 1}{2}\right]$$

Proof. Let k be any natural number and P_0 a $u - v$ geodesic in G. Let us form a new graph G_1 by removing all the edges of P_0. Set $d_1(u, v)$ equal to the length of a shortest $u - v$ path in G_1. Then the inequality

$$d_1(u, v) \geqq d(u, v)$$

holds.

Let P_1 be a $u - v$ geodesic in G_1. Then we obtain the graph G_2 by removing all edges of P_1 from G_1. Clearly

$$d_2(u, v) \geqq d_1(u, v) \geqq d(u, v).$$

We continue in this manner until we obtain a graph G_r with $u - v$ geodesic P_r of the length $\leqq k$ but $u - v$ geodesic P_{r+1} is of the length $> k$. Denote this graph G' we denote $d'(u, v)$ and similary $B_n'(u, v)$, $H_n'(u, v)$ respectively. It can be shown that

$$d'(u, v) \geqq k + 1$$

In this manner we have removed from $\bar{G}$ r edge-disjoint $u-v$ paths and then for G'

$$B'_n(u, v) \leqq B_n(u, v) - r$$

holds.

The minimal number of the edges of G remowing of which destrosys all $u-v$ paths with the length $\leqq n$ we can estimate in the following way

$$H_n(u, v) = H'_n(u, v) + r.k$$

If G' is connected then by Theorem 1 we have

$$H'_n(u, v) \leqq (m+1)B'_n(u, v)$$

where $m = n - d'(u, v)$.

So

$$H'_n(u, v) \leqq (n - d'(u, v) + 1)B'_n(u, v) \leqq$$
$$\leqq (n - (k+1) + 1)B'_n(u, v) \leqq (n-k)B'_n(u, v)$$

Hence

$$H_n(u, v) \leqq (n-k)B'_n(u, v) + r.k \leqq (n-k)(B_n(u, v) - r) + r.k =$$
$$= (n-k)B_n(u, v) + r.(2k - n)$$

For $k = \left[\frac{n}{2}\right]$ we have $r.(2k-n) \leqq 0$, and the statement of Theorem 2 follows.

Now let G' be disconnected. Then

$$H'_n(u, v) = B'_n(u, v) = 0$$

and the statement trivialy holds.

q.e.d.

The estimation from Theorem 2 is sharp for $n = 2, 4$ but it is not for $n = 3$ as follows from Theorem 3.

Theorem 3. For any connected graph G with distinct vertices u, v we have

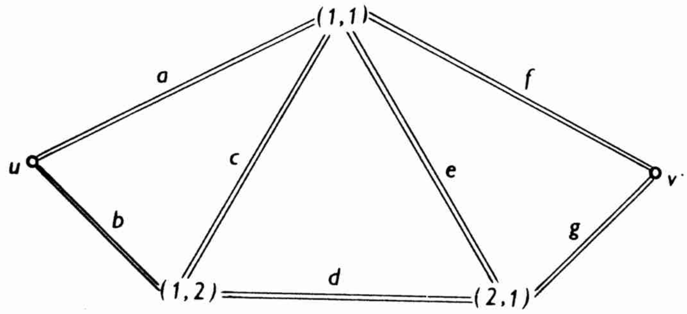
$$H_3(u, v) = B_3(u, v).$$

Proof. We divide all vertices of G except for u, v into classes as follows. The vertex w belongs to (i, j) if

$$d(u, w) = i \quad \text{and} \quad d(w, v) = j.$$

As we are interested only in the paths of the length not greater than 3, we omit all classes (i, j) with $i + j > 3$. The scheme of such a graph follows from *Fig. 2.* We denote it $\hat{G}$.

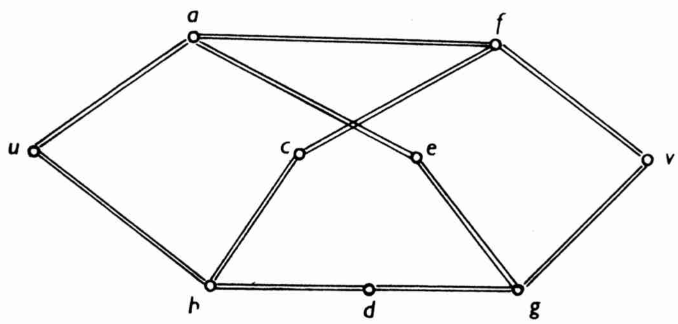
We form a new graph $\hat{G}_1$ in the following way. A set of vertices of $\hat{G}_1$ is the set



of edges of $\hat{G}$ and vertices u, v. The vertex u (or v) is adjacent to a vertex z iff the vertex u (or v) is incident with corresponding edge in $\hat{G}$. Two vertices x and y or $\hat{G}_1$ are adjacent if the corresponding edges in $\hat{G}$ are adjacent and both of them belong to a $u-v$ path with the length not greater than 3. The schema of such a graph is shown in *Fig. 3.* From this construction immediately follows

(1) There is one to one correspondence between edges of G and vertices of G_1 except for vertices u, v,

(2) Every $u-v$ path in $\hat{G}$ with length $\leqq 3$ corresponds to exactly one $u-v$ path in $\hat{G}_1$ with length $\leqq 4$ and conversly.



Denote by $A_4^{\hat{G}_1}(u, v)$ the maximal number of disjoint paths in $\hat{G}_1$. One can easy prove that

$$A_4^{\hat{G}_1}(u, v) = B_3^{\hat{G}}(u, v).$$

Denote $V_4^{\hat{G}_1}(u, v)$ the minimal number of vertices which deletion destroys all paths of length not exeeding 4. Then we have

$$V_4^{\hat{G}_1}(u, v) = H_3^{\hat{G}}(u, v)$$

From Theorem 3 of [1] we have

$$V_4^{G_1}(u, v) = A_4^{G_1}(u, v)$$

hence

$$B_3^{G}(u, v) = H_3^{G}(u, v)$$

and the proof follows.

q.e.d

3. Some constructions

In oder to find any lower bound for the maximum of the ratio

$$\frac{H_n(u, v)}{B_n(u, v)}$$

for any graph we form some constructions.

Let L be a $u - v$ path of the length $n - t$. Denote its vertices $u = x_0, x_1, x_2, \ldots, x_{n-t} = v$. Now we add $(n-t).t$ new vertices divided into $n-t$ classes with t vertices. Now we construct a graph G_n in the following way. Every vertex of i-th class is adjacent to vertices x_{i-1} and x_i of the path L and the graph G_n does not contain any more edges except the mentioned above. Any vertex added to the path L with incident edges we call a "roof". so we have t roofs over every edge of L.

Take any $u - v$ path P in G_n of the length not exeeding n, such that it contains exactly r roofs. For its length $d(P)$ we have

$$d(P) = n - t - r + 2r \leqq n$$

that means $r \leqq t$.

Let P_1 and P_2 be any $u - v$ path of G_n with not more than $\frac{n-t}{2}$ roofs, then there exists at least one edge common for P_1 and P_2 lying on the path L. So in the case

$$t \leqq \left[\frac{n}{3}\right]$$

we have $B_n(u, v) = 1$.

Now we show that in G_n we have

$$H_n(u, v) = t + 1.$$

Suppose $H_n(u, v) \leqq t$. That means there exist t vertices the removing of which destroys all $u - v$ paths with the length $\leqq n$. Without loosing generality we can suppose that all t-edges are members of the path L. There exists an $u - v$ path P' passing these t-edges across the roofs. For its length we have

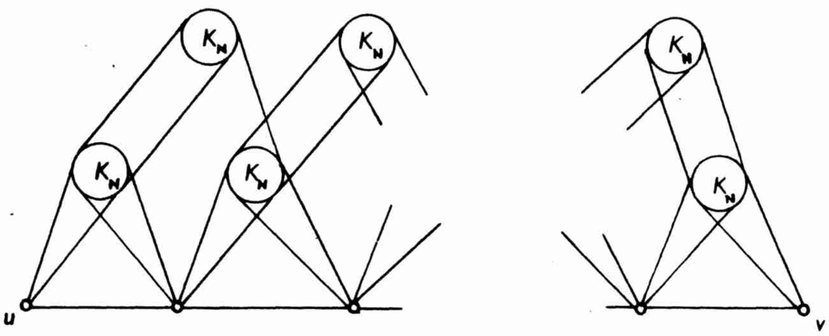
$$d(P') = d(L) - t + 2t = n$$

That means deleting of t-edges is not enough to destroy all $u - v$ paths with the length $\leqq n$.
For the ratio we have

$$\frac{H_n(u, v)}{B_n(u, v)} = \left[\frac{n}{3}\right] + 1$$

The graph G_n we have constructed has the edge connectivity $\lambda(G_n) = 2$ and the connectivity $\varkappa(G_n) = 1$. We form a graph $G_{n,k}$ with greater connectivity and edge connectivity, respectively.

Let $k \geqq 2$ be an arbitrary integer. Set $N = \max\{k - 1, t\}$. In the graph G_n we replace t roofs lying over any edge of G_n by a complete graph K_N. Every vertex of i-th copy of K_N is adjacent to both vertices x_i and x_{i+1}, respectively. Add $n - (t + 1)$ new copies of the graph K_n in such a way that all vertices of i-th copy of K_N and $i + 1$-th copy are adjacent to all vertices of a new copy of K_N. See *Fig. 4.*



From the construction of the graph $G_{n,k}$ one can see that connectivity and edge connectivity of the graph $G_{n,k}$ is at least k and the inequality

$$\frac{H_n(u, v)}{B_n(u, v)} \leqq \left[\frac{n}{3}\right] + 1$$

holds.

From the above construction and Theorem 2 we have

$$\left[\frac{n}{3}\right]+1 \leqq \sup_G \frac{H_n^G(u, v)}{B_n^G(u, v)} \leqq \left[\frac{n+1}{2}\right]$$

where G is an arbitrary finite graph.

The result of [1]

$$\sup \frac{V_n(u, v)}{A_n(u, v)} \geqq \sqrt{\frac{n}{2}}$$

cannot be improved using the construction of $G_{n,k}$ as this graph is not a line graph of any graph.

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SÚHRN

O JEDNOM ZOVŠEOBECNENÍ MENGEROVEJ VETY

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Práca pojednáva o vzťahu medzi počtom disjunktných ciest dĺžky $\leqq n$ v grafe, ktoré spájajú dva pevne zvolené vrcholy a minimálnym počtom hrán, ktoré treba vynechať na ich prerušenie. Výsledky sú presné pre hodnoty $n = 1, 2, \ldots, 5$. V ostatných prípadoch sú uvedené horné aj dolné odhady.

РЕЗЮМЕ

ОБ ОДНОМ ОБОБЩЕНИИ ТЕОРЕМЫ МЕНГЕРА

Льудовит Ниепел, Даниела Шафаржикова, Братислава

В работе изучается отношение между числом непересекающихся простых цепей длины $\leqq n$ соединяющих две вершины графа и минимальным числом ребер, которые нужно отбросить для разделения этих цепей. Предлагаемые результаты точные для $n = 1, 2, \ldots, 5$.