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POINT-MAXIMAL SUBGRAPHS WITH A GIVEN DIAMETER

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Under a graph we mean throughout the paper an undirected finite graph without loops and multiple lines. Our terminology as well as denotation is based on [1] except for the given here. A graph $G=(V, E)$ has the point set $V=V(G)$ and the set of lines $E=E(G)$. We denote by $d(G)$ the diameter of G. For a given graph G, the point-maximal subgraph H with a given diameter d is defined as a subgraph of G with the following two properties:

a) $d(H)=d$

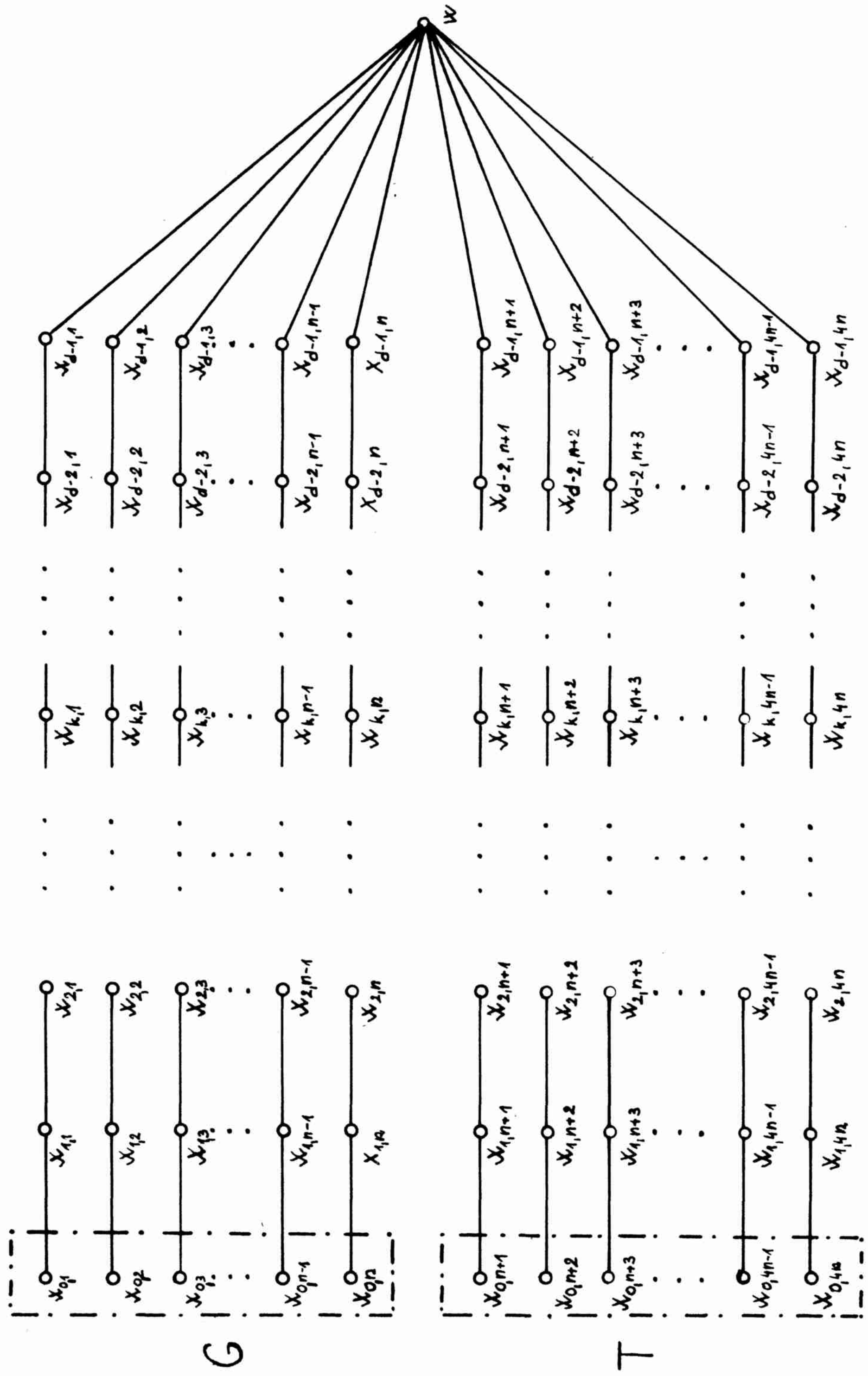
b) $|V(R)|\leqslant|V(H)|$ for every subgraph R of G with $d(R)=d$.

Given a graph G, the problem investigated in this article is to find a point-maximal subgraph of G, with a given diameter d. This problem for $d=1$ belongs into the class of NP-complete problems (see [2]). As we show in this paper the problem is NP-complete for an arbitrary given $d\geqslant 1$, too.

To prove this, let us construct for a given graph $G=(V, E)$ with $V(G)=\{x_{0,1}, x_{0,2}, \ldots, x_{0,n}\}$ a new graph G' (see *Fig. 1*) as follows (the idea of this construction is based on [3]):

$$V(G')=V(G)\cup\{x_{0,n+i}\mid i=1,2,\ldots,3n\}\cup$$
$$\cup\{x_{i,j}\mid i=1,2,\ldots,d-1;\ j=1,2,\ldots,4n\}\cup\{w\}.$$
$$E(G')=E(G)\cup\{x_{0,n+i}x_{0,n+j}\mid i\neq j;\ i,j=1,2,\ldots,3n\}\cup$$
$$\cup\{x_{0,n+i}x_{0,j}\mid i=1,2,\ldots,3n;\ j=1,2,\ldots,n\}\cup$$
$$\cup\{x_{i,j}x_{i+1,j}\mid i=0,1,\ldots,d-2;\ j=1,2,\ldots,4n\}\cup$$
$$\cup\{x_{d-1,j}w\mid j=1,2,\ldots,4n\}.$$

Let $\mathcal{M}$ be a set of all point-maximal complete subgraphs of G. The induced (complete) subgraph of G' with the set $\{x_{0,n+i}\mid i=1,2,\ldots,3n\}$ is denoted by T. Let $P=V(G)\cup V(T)$. Let $\mathcal{S}$ denote the set of all induced subgraphs $\langle M\rangle$ of the graph G' with:


w
G
T

$$M = V(H) \cup V(T) \cup \{x_{i,j} \mid i = 1, 2, \ldots, d-1; j = k_1, k_2, \ldots, k_s \quad (1)$$
where k_r $(r = 1, 2, \ldots, s)$ are exactly all the indices for which $x_{0,k_r} \in V(H)\} \cup \{w\}$,
where $H \in \mathcal{M}$.

Lemma. Let $R \in \mathcal{S}$. Then the inequality $|V(S)| < |V(R)|$ holds for an arbitrary subgraph S of G' such, that $d(S) = d$ and $S \notin \mathcal{S}$.

Proof. Since $R \in \mathcal{S}$, there is $H \in \mathcal{M}$ such, that $R \equiv \langle M \rangle$ for M defined by (1). It follows from the definition of G' that $d(R) = d$ and $|V(R)| = (3n+h)d + 1$, where $h = |V(H)|$. Let S be a subgraph of G' for which the assumptions of lemma hold. We will distinguish the next two cases:

1. $\qquad w \in V(S).$

a) $P \cap V(S) \neq \emptyset$.

Then $x_{i,j} \in V(S)$ if and only if $x_{0,j} \in V(S)$ for every $j \in \langle 1, 4n \rangle$, $i \in \langle 1, d-1 \rangle$. Let $x_{0,j}$, $x_{0,k}$ belong to $P \cap V(S)$, $x_{0,j} \neq x_{0,k}$ and $x_{0,j}x_{0,k} \notin E(G')$. Then $d_S(x_{d-1,k}, x_{0,j}) > d$, a contradiction. Since $S \notin \mathcal{S}$, then also $\langle P \cap V(S) \rangle \notin \mathcal{M}$, and we have:

$$|P \cap V(S)| < 3n + h.$$

From this it follows immediately that: $|V(S)| < (3n+h)d + 1$.

b) $P \cap V(S) = \emptyset$.

Let

$$r = \max_{i,j} d_S(x_{i,j}, w) \; i \in \langle 1, d-1 \rangle, j \in \langle 1, 4n \rangle, x_{i,j} \in V(S).$$

Then

$$|V(S)| \leq r + 1 + p(4n-1), \text{ where } r \leq d, \; r + p = d \text{ and } p \leq \left[\frac{d}{2}\right].$$

For $r = d$ we have $|V(S)| = d + 1 < (3n+h)d + 1$.
Let $r < d$. Then

$$|V(S) < d + 1 + \left[\frac{d}{2}\right](4n-1) < (3n+h)d + 1 \text{ for } d \geq 2.$$

2. $\qquad w \notin V(S).$

Since $d(S) = d$, the set $P \cap V(S)$ cannot be empty. Let $r = \max_{i,j} d_S(P, x_{i,j})$, $i \in \langle 1, d-1 \rangle$, $j \in \langle 1, 4n \rangle$ and $x_{i,j} \in V(S)$. Then

$$|V(S)| \leq r + 4n + (4n-1)p \text{ where } p \text{ is an integer such}$$

$$p \leq \left[\frac{d}{2}\right] \qquad \text{if } d \text{ is odd}$$

that

$$p \leq \left[\frac{d}{2}\right] - 1 \quad \text{if } d \text{ is even}$$

Thus we see

$$|V(S)| \leq r + 4n + (4n-1)p \leq d + 4n + (4n-1)\left[\frac{d}{2}\right] <$$
$$< (h+3n)d + 1 \quad \text{if } d \text{ is odd, } d \geqq 3,$$

and

$$|V(S)| \leq r + 4n + (4n-1)\left(\left[\frac{d}{2}\right] - 1\right) = d + 1 + (4n-1)\left[\frac{d}{2}\right] < (h+3n)d + 1$$
$$\text{if } d \text{ is even, } d \geq 2.$$

Hence $|V(R)| = (3n+h)d + 1 > |V(S)|$.
This completes the proof.

Corollary. A subgraph R of the graph G' is point-maximal with the diameter d if and only if R belongs to $\mathcal{S}$.

Proof. Since $d(R) = d$, for every $R \in \mathcal{S}$, the assertion follows immediately from the previous lemma.

Theorem. Given a point-maximal complete subgraph in a graph G, a point-maximal subgraph of the diameter d in G' can be determined in a polynomial number of steps and vice versa.

Proof. From the proof of lemma it follows:

a) If $H \subseteq G$ is a point-maximal subgraph in G, then the subgraph of G' defined by (1) is point-maximal.

b) If S is a point-maximal subgraph with the diameter d in G', then a point-maximal complete subgraph in a graph G is defined as an induced subgraph on the set $V(S) \cap V(G)$.

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SÚHRN

VRCHOLOVO-MAXIMÁLNE PODGRAFY S DANÝM PRIEMEROM

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V práci je dokázané, že problém najdenia vrcholovo maximálneho podgrafu s daným priemerom v danom grafe patrí do skupiny *NP*-úplných problémov.

РЕЗЮМЕ

ВЕРШИННО-МАКСИМАЛЬНЫЕ ПОДГРАФЫ ДАННОГО ДИАМЕТРА

Петер Кыш, Алойз Ваврух, Братислава

В работе доказано, что проблема нахождения вершинно-максимального подграфа с данным диаметром в данном графе принадлежит к категории *NP*-полных задач.