UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLII—XLIII—1983

ON PALINDROMIC NUMBERS

HELENA KRESOVÁ, TIBOR ŠALÁT, Bratislava

Let g be an integer, $g \geqq 2$. The expression of the positive integer a in the scale of g:

$$a = c_n g^n + c_{n-1} g^{n-1} + \ldots + c_0$$

($n \geqq 0$, c_k are integers, $0 \leqq c_k < g$, $k = 0, 1, \ldots, n$, $c_n \neq 0$) will be shortly written as

$$a = (c_n c_{n-1} \ldots c_0) g \tag{1}$$

The number a (see (1)) is said to be a palindromic number (in the scale of g) if

$$(c_n c_{n-1} \ldots c_0) g = (c_0 c_1 \ldots c_n) g$$

Hence if a is a palindromic number, then $c_0 \neq 0 \neq c_n$ (cf. [2], p. 216).

E.g. the numbers 121,2332 are palindromic (in the scale of 10).

In [1] (p. 63) the following generalization of the scale of g is introduced:

Let $\{q_k\}_{k=1}^{\infty}$ be a sequence of positive integers, $q_k \geqq 2$ ($k = 1, 2, \ldots$). Then each positive integer a can be uniquely expressed in the form

$$a = c_n . q_1 \ldots q_n + c_{n-1} . q_1 \ldots q_{n-1} + \ldots + c_1 . q_1 + c_0, \tag{2}$$

where $n \geqq 0$, c_k ($k = 0, 1, \ldots, n$) are integers, $0 \leqq c_k < q_{k+1}$ ($k = 0, 1, \ldots, n$), $c_n \neq 0$.

The equality (2) is said to be the expression of the number a in the scale of $Q = q_1, q_2, \ldots$ and it will be written in the form

$$a = (c_n c_{n-1} \ldots c_0) Q \tag{3}$$

Putting $q_k = q \geqq 2$ ($k = 1, 2, \ldots$) we get the scale of g.

The notion of palindromic numbers can be extended for the scales of $Q = q_1, q_2, \ldots$. The positive integer a (see (3)) will be called a palindromic number in the scale of $Q = q_1, q_2, \ldots$ if

$$(c_n c_{n-1} \ldots c_0) Q = (c_0 c_1 \ldots c_n) Q$$

Hence, if a is a palindromic number in the scale of $Q = q_1, q_2, \ldots$, then $c_0 \neq 0 \neq c_n$.

In this paper we shall prove some elementar results on the set of all palindromic numbers in the given scale.

Theorem 1. Let

$$P=\{p_1<p_2<\dots\}$$

be the set of all palindromic numbers in the scale of $Q=q_1, q_2, \dots$. Then we have

$$\sum_{k=1}^{\infty} p_k^{-1}<+\infty$$

Proof. Denote by P_j $(j=1, 2, \dots)$ the set of all palindromic numbers a in the scale of $Q=q_1, q_2, \dots$ for which

$$q_1 \dots q_j \leqq a < q_1 . q_2 \dots q_j . q_{j+1} \tag{4}$$

Each such a number has the form

$$a=c_j . q_1 \dots q_j + c_{j-1} . q_1 \dots q_{j-1} + \dots + c_0, \tag{5}$$

$$0 \leqq c_k < q_{k+1} \ (k=0, 1, \dots, j),\ c_j \neq 0 .$$

Further $c_0=c_j$, $c_1=c_{j-1}$, $\dots$, $c_{[\frac{j}{2}]}=c_{j-[\frac{j}{2}]}$. From this we get easily the following eatimatimation for the number $|P_j|$ of all palindromic numbers of the form (4)

$$|P_j| \leqq q_1 . q_2 \dots q_{[\frac{j}{2}]+1} \tag{6}$$

From (4), (6) we get

$$A_j=\sum_{a\in P_j} a^{-1} \leqq \frac{q_1 \dots q_{[\frac{j}{2}]+1}}{q_1 \dots q_j}=$$

$$=\frac{1}{q_j . q_{j-1} \dots q_{[\frac{j}{2}]+2}} \leqq \frac{1}{2^{j-[\frac{j}{2}]-1}} \leqq \frac{2}{(\sqrt{2})^j}$$

Hence

$$\sum_{j=1}^{\infty} A_j<+\infty \tag{7}$$

Since

$$\sum_{k=1}^{\infty} p_k^{-1} \leqq \sum_{k=1}^{q_1-1} k^{-1}+\sum_{j=1}^{\infty} A_j,$$

the assertion follows on the basis of (7).

Let $A \subset N=\{1, 2, \dots\}$, denote by $A(n)$ the number of all $a \in A$ with $a \leqq n$. If there exists

$$h(A)=\lim_{n\to\infty}\frac{A(n)}{n},$$

then this number is called the asymptotic density of the set A (cf. [6], p. 100; [7]). It is a well-known fact that if $\sum_{a\in A} a^{-1}<+\infty$, then $h(A)=0$ (cf. [3]; [5], p. 100; [6]). Hence we get from Theorem 1 the following result.

Corollary. Let P have the same meaning as in Theorem 1. Then we have $h(P)=0$.

In what follows we shall show that in the case of the scale of $g\geqq 2$ it is possible to determine the exponent of convergence (cf. [4], p. 40) of the sequence of all palindromic numbers. We shall show that this exponent of convergence does not depend on the number g.

Theorem 2, Let g be an integer, $g\geqq 2$, let

$$p_1<p_2<\ldots \tag{7}$$

be the sequence of all palindromic numbers in the scale of g. Then the exponent of convergence of the sequence (7) is equal to 1/2.

Proof. Denote by P_n the set of all palindromic numbers a in the scale of g for which

$$g^n\leqq a<g^{n+1} \qquad (n=1, 2, \ldots).$$

Each $a\in P_n$ has the expression (in the scale of g)

$$a=c_n\,.\,g^n+c_{n-1}\,.\,g^{n-1}+\ldots+c_0, \tag{8}$$
$$0\leqq c_k<g \qquad (k=0, 1, \ldots, n), \qquad c_n\neq 0.$$

Similarly as in the proof of the foregoing theorem it can be showed that the estimation

$$|P_n|\leqq g^{\frac{n}{2}+1} \tag{9}$$

holds for the number $|P_n|$ of elements of the set P_n. But then we have (for $\sigma>0$) on the basis of (8) and (9)

$$B_n=\sum_{a\in P_n} a^{-\sigma}\leqq\frac{g}{g^{n(\delta-\frac{1}{2})}}$$

Hence for $\sigma>1/2$ we have $\sum_{n=1}^{\infty} B_n<+\infty$ and since

$$\sum_{k=1}^{\infty} p_k^{-\sigma}=\sum_{k=1}^{g-1} k^{-\sigma}+\sum_{n=1}^{\infty} B_n,$$

we see that the series

$$\sum_{k=1}^{\infty} p_k^{-\sigma} \tag{10}$$

converges for $\sigma > 1/2$.

We shall show that the series (10) diverges for $0 < \sigma \leqq 1/2$.

It can be easily verify that if n is an even number, $n = 2s$ $(s \geqq 1)$, then

$$|P_n| \geqq (g-1) \,.\, g^s = (g-1) \,.\, g^{\frac{n}{2}}$$

and if n is an odd number, $n = 2s+1$ $(s \geqq 0)$, then

$$|P_n| \geqq (g-1) \,.\, g^s = (g-1) \,.\, g^{\frac{n}{2}-\frac{1}{2}}$$

Hence for every n we have

$$|P_n| \geqq (g-1) \,.\, g^{\frac{n}{2}-1} \tag{11}$$

On account of (11) we obtain for $\sigma > 0$:

$$\sum_{a \in P_n} a^{-\sigma} \geqq \frac{g-1}{g^{\sigma+1}} \cdot g^{n(\frac{1}{2}-\sigma)}$$

From this it can be easily chacked that the series (10) diverges for $\sigma \leqq 1/2$.

With respect to the foregoing facts it follows from the definition of the exponent of convergence (cf. [4], p. 40) that the exponent of convergence of the sequence (7) is equal to 1/2. This ends the proof.

The foregoing theorem enables us to give a simple estimation from below for the n-th term of the increasing sequence of all palindromic numbers in the scale of g.

Theorem 3. Let g be an integer, $g \geqq 2$, let

$$p_1 < p_2 < \dots$$

be the sequence of all palindromic numbers in the scale of g. Then for each $\varepsilon > 0$ there exists such an n_0 that for each $n > n_0$ we have

$$p_n > n^{\frac{1}{2}-\varepsilon}$$

Proof. On account of a well-known formula for the exponent of convergence (cf. [4], p. 40) it follows from Theorem 2 that

$$\frac{1}{2} = \limsup_{n \to \infty} \frac{\log n}{\log p_n} \tag{12}$$

Choose $\eta > 0$ such that

$$\left(\frac{1}{2}+\eta\right)^{-1}>\frac{1}{2}-\varepsilon \tag{13}$$

Then on account of (12) there exists such an n_0 that for each $n>n_0$ we have

$$\frac{\log n}{\log p_n}<\frac{1}{2}+\eta$$

From this we get

$$p_n>n^{\frac{1}{1/2+\eta}}$$

and so owing to (13) we obtain

$$p_n>n^{\frac{1}{2}-\varepsilon}$$

This ends the proof.

REFERENCES

[1] Erdős, P.—Surányi, J.: Válogatott fejezetek a számelméletből. Tankönyvkiadó vállalat, Budapest, 1960.

[2] Kirch, A. M.: Elementary number theory: a computer approach. Intext educational publishers, New York—London, 1974.

[3] Krzyś, J.: Twierdzenie Oliviery i jego uogólnienia. Prace matem. 2 (1956), 159—164.

[4] Pólya, G.—Szegő, G.: Aufgaben und Lehrsätze aus der Analysis I. (Russian translation). Nauka, Moskva, 1978.

[5] Šalát, T.: Nekonečné rady. Academia, Praha, 1974.

[6] Šalát, T.: Rad prevrátených hodnôt všetkých prvočísel a niektoré výsledky o konvergencii čiastočných radov harmonického radu. Pokroky matematiky, fyziky a astr. 10 (1956), 168—178.

Addresses of authors:

Received: 3. 6. 1980

Tibor Šalát
Katedra algebry a teórie čísel MFF UK
Pavilón matematiky,
Mlynská dolina,
842 15 Bratislava,
Czechoslovakia.

Helena Kresová
Stredná priemyselná škola v Partizánskom
958 01 Partizánske

РЕЗЮМЕ

О ПАЛИНДРОМИЧЕСКИХ ЧИСЛАХ

Елена Кресова, Тибор Шалат, Братислава

Пусть

$$a = c_n g^n + \ldots + c_0 = (c_n \ldots c_0)g$$

выражение натурального числа а в системе счисления с основанием $g \geqq 2$. Число a называется палиндромическим (в системе счисления с основанием g) если

$$(c_n \ldots c_0)g = (c_0 \ldots c_n)g$$

В работе доказано что показатель сходимости последовательности всех палинбромических чисел (в системе счисления с основанием g) равный числу 1/2 для всякого $g \geqq 2$.

SÚHRN

O PALINDROMICKÝCH ČÍSLACH

Helena Kresová a Tibor Šalát, Bratislava

Nech

$$a = c_n \cdot g^n + \ldots + c_0 = (c_n \ldots c_0)g$$

je vyjadrenie prirodzeného čísla a v g-adickej sústave, $g \geqq 2$. Číslo a sa nazýva palindromickým číslom (v g-adickej sústave), ak

$$(c_n \ldots c_0)g = (c_0 \ldots c_n)_g.$$

V práci je dokázané, že exponent konvergencie postupnosti všetkých palindromických čísel (v g-adickej sústave) sa rovná 1/2 pre každé $g \geqq 2$.

UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLII—XLIII—1983

ON PROPERTIES OF ARCHIMEDEAN ORDERED FIELDS THAT ARE EQUIVALENT TO THE COMPLETENESS

TIBOR ŠALÁT, ANNA TARABOVÁ, Bratislava

The notations and definitions of fundamental notions used in this paper are taken from [2] and [3].

The ordered field $\langle A, +, ., < \rangle$ is said to be complete if each fundamental sequence of elements from A is convergent in A. The ordered field $\langle A, +, ., < \rangle$ is said to be continuously ordered if the ordered set $(A, <)$ is continuously ordered, i.e. if every non-empty subset of A which is bounded from above has the least upper bound in A.

It is well-known that the ordered field $\langle R, +, ., < \rangle$ of all real numbers can be characterized as an Archimedean complete field. This fact guarantees the importance of the study of ordered fields for the elementary mathematics in secondary schools and theoretic arithmetics.

In [2] (pp. 95—102) seven properties of ordered fields are given (these properties are denoted by I—VII) that are mutually equivalent. It follows from these equivalences that an ordered field $\langle A, +, ., < \rangle$ is continuously ordered if and only if it is Archimedean and complete. In this paper we shall give some further properties of ordered fields that are equivalent to the property of continuous ordering (or to the arbitrary property from the properties I—VII on pp. 95—102 of [2]).

Definition 1. Let $\langle A, +, ., < \rangle$ be an ordered field. The set $M \subset A$ is said to be a compact set if every sequence of elements from M contains a convergent subsequence, the limit of which belongs to M.

Theorem 1. An Archimedian field $\langle A, +, ., < \rangle$ is complete if and only if every closed and bounded set $M \subset A$ is compact.

For the proof of Theorem 1 we shall use the following two auxiliary results.

The following Lemma 1 is a simple generalization of a result from [1].

Let us remark that a sequence $\{a_n\}_{n=1}^{\infty}$ of elements of an ordered set A is said to be monotone if it is nondecreasing i.e. if

$$a_1 \leqq a_2 \leqq \ldots \leqq a_n \leqq a_{n+1} \leqq \ldots$$

or if it is nonincreasing i.e. if

$$a_1 \geqq a_2 \geqq \ldots \geqq a_n \geqq a_{n+1} \geqq \ldots .$$

Lemma 1. Every sequence of elements of an ordered set contains a monotone subsequence.

Proof. Let

$$\{x_n\}_{n=1}^{\infty} \tag{1}$$

be a sequence of elements of an ordered set A. Denote by N the set of all positive integers. Let P be the set of all $n \in N$ with the following property: there is an n_0 (depending on n) such that for each $i > n_0$ we have $x_i \geqq x_n$. Similarly denote by Q the set of all $n \in N$ with the property: there is an m_0 (depending on n) such that for each $i > m_0$ we have $x_i \leqq x_n$. Put $S = N - (P \cup Q)$. Then at least one of the sets S, P, Q is infinite.

Let the set P be infinite, let

$$P = \{p_1 < p_2 < \ldots < p_k < \ldots\} .$$

Put $n_1 = p_1$. Since $n_1 \in P$, we have $x_{n_1} \leqq x_{p_j}$ for all sufficiently large $j \geqq 2$. Denote by j_0 the minimal number from these j's. Put $n_2 = p_{j_0}$. Hence $x_{n_1} \leqq x_{n_2}$, $n_2 \in P$, a.s.o. So by induction we can construct a sequence

$$x_{n_1} \leqq x_{n_2} \leqq \ldots \leqq x_{n_k} \leqq \ldots$$

that is a monotone subsequence of (1).

If the set Q would be infinite, then we could construct a nonincreasing subsequence of (1).

If both P, Q are finite sets, then there exists such an m that

$$\{m+1, m+2, \ldots, m+k, \ldots\} \subset S .$$

Put $n_1 = m+1$. Since $n_1 \notin P \cup Q$, there is an infinite number of i's with $x_{n_1} < x_i$ and simultaneously an infinite number of i's with $x_{n_1} > x_i$. Using a consideration which is similar to the previous consideration we can construct in this case a decreasing subsequence of (1) and also an increasing subsequence of (1).

Lemma 2. Every monotone bounded sequence of elements of an Archimedean ordered field is fundamental.

Proof. Let $\langle A, +, \cdot, < \rangle$ be an Archimedian field, let

$$a_1 \leqq a_2 \leqq \ldots \leqq a_k \leqq a_{k+1} \leqq \ldots \tag{2}$$

be a sequence of elements of A (if we would have

$$a_1 \geqq a_2 \geqq \ldots \geqq a_k \geqq a_{k+1} \geqq \ldots ,$$

then the proof would run in a similar way).

According to the assumption there exists such an $a \in A$ that $a_n \leqq a$ $(n = 1, 2, \ldots)$.

Construct the sequence

$$0_A < 1_A \leqq a_2 - a_1 + 1_A \leqq a_3 - a_1 + 1_A \leqq \ldots \tag{3}$$

Let $e \in A$, $e > 0_A$. Choose an $s \in N$ such that $1/s < e$. There exists such s because A is Archimedean. From the same reason there exists such an $n \in N$ that

$$n \frac{1}{s} \geqq a - a_1 + 1_A \tag{4}$$

Denote by V the set of all $k \in N$ for which

$$a_n - a_1 + 1_A \leqq k \frac{1}{s} \qquad (n = 1, 2, \ldots).$$

On account of (4) we have $V \neq \emptyset$. Hence there exists min $V = p$. By the definition of p there exists such $j \in N$ that

$$a_j - a_1 + 1_A > (p-1) \frac{1}{s} \tag{5}$$

For arbitrary numbers $k, i > j$ (let e.g. $k \geqq i$) we get from (5) and the definition of p

$$(p-1) \frac{1}{s} < a_i - a_1 + 1_A \leqq a_k - a_1 + 1_A \leqq p \frac{1}{s}$$

From this by the choice of s we obtain $|a_i - a_k| < e$. Hence (2) is a fundamental sequence.

Proof of Theorem 1. Let $\langle A, +, ., < \rangle$ be an Archimedean ordered complete field. Let $M \subset A$ be a bounded and closed set in A and $x_n \in M$ $(n = 1, 2, \ldots)$. According to Lemma 1 there exists a monotone subsequence

$$\{x_{n_k}\}_{k=1}^{\infty} \tag{6}$$

of the sequence $\{x_n\}_{n=1}^{\infty}$. It follows from Lemma 2 that the sequence (6) is fundamental. Since the field A is complete, the sequence (6) converges. Since the terms of the sequence (6) belong to the closed bounded set M, the limit of (6) belongs to M, too. Hence M is a compact set.

2. Let every bounded and closed subset of the field A be compact. Let

$$\{y_n\}_{n=1}^{\infty} \tag{7}$$

be a fundamental sequence of elements of A. Then (7) is bounded ([2], p. 73). Therefore there exists such $b \in A$ that

$$|y_n| \leqq b \qquad (n = 1, 2, \ldots).$$

It is easy to verify that the interval $[-b, b]$ is a closed bounded set, therefore according to the assumption this interval is a compact set. Since y_n $(n=1, 2, \ldots)$ belong to $[-b, b]$, there exists a convergent subsequence of (7). But it is a well-known fact that the fundamental sequence is convergent if it has a convergent subsequence ([3], p. 41). Hence (7) is a convergent sequence. This ends the proof.

In what follows denote by T the topology on A (A is an ordered field) the open basis of which is the system of all open intervals (a, b) with $a, b \in A$.

Theorem 2. An Archimedean field $\langle A, +, ., < \rangle$ is complete if and only if the topological space (A, T) is connected.

Proof. 1. Let the Archimedean field $\langle A, +, ., < \rangle$ be complete. Let us assume that the space (A, T) is disconnected. Then we have

$$A = X \cup Y, \ X \cap Y = \emptyset, \ X \neq \emptyset \neq Y,$$

X, Y are open (and simultaneously closed) subsets of A.

Let us choose $x_1 \in X$, $y_1 \in Y$ and e.g. let $x_1 < y_1$ (in the contrary case, i.e. if $y_1 < x_1$, the proof runs analogously).

Let us construct the element $(x_1 + y_1)/2 \in A$. If this element belongs to X (belongs to Y), then we put $x_2 = (x_1 + y_1)/2$, $y_2 = y_1 (x_2 = x_1, y_2 = (x_1 + y_1)/2)$,

a.s.o. So by induction we construct two sequences $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ of elements of A with the following properties:

$$x_k \in X, \ y_k \in Y \qquad (k=1, 2, \ldots), \tag{8}$$

$$x_1 \leqq x_2 \leqq \ldots \leqq x_k \leqq \ldots,$$

$$x_k \leqq y_1 \qquad (k=1, 2, \ldots),$$

$$y_k - x_k = \frac{y_1 - x_1}{2^{k-1}} \qquad (k=1, 2, \ldots). \tag{9}$$

Hence the sequence $\{x_k\}_{k=1}^{\infty}$ is monotone and bounded. According to Lemma 2 this sequence is fundamental. On account of the completeness of A there exists

$$y = \lim_{k \to \infty} x_k \tag{10}$$

Since A is Archimedean, we have from (9), (10)

$$y = \lim_{k \to \infty} y_k \tag{10'}$$

But X, Y are closed sets, hence on the basis of (8), (10), (10′) we get $y \in X \cap Y$ contrary to the disjointness of X, Y.

2. Let $\langle A, +, ., < \rangle$ be an Archimedean field which is not complete. Then A is not continuously ordered ([2], p. 95). Therefore there is such a non-empty set $M \subset A$ which is bounded from above and has not the least upper bound in A.

Denote by X the set of all upper bounds of M and put $Y = A - X$. Then $X \neq \emptyset$ and since $M \neq \emptyset$, we have $Y \neq \emptyset$, too.

We shall prove that each of the sets X, Y is open in the space (A, T).

Let $x \in X$. Then x is an upper bound for M and since M has not the least upper bound, there exists such an $a \in A$ that $a \in X$, $a < x$. But then we have

$$x \in (a, x + 1_A) \subset X$$

and hence x is an interior point of X. Therefore X is an open set.

Let $y \in Y$. According to the definition of Y there exists such an $x_0 \in M$ that $y < x_0$. But then we have

$$y \in (y - 1_A, x_0) \subset Y,$$

hence y is an interior point of Y and so Y is an open set This ends the proof.

In the formulation of Theorems 1 and 2 there were situated such properties of Archimedean fields that had been equivalent to the completeness of A and had been formulated using subsets of A. The further properties that will be introduced in the following two theorems, will be equivalent to the completeness of A and they will be formulated using the functions of the type $f: B \to A$, where $B \subset A$.

Let $f: B \to A$, $B \subset A$, where $\langle A, +, ., < \rangle$ is an ordered field. The function f is said to be continuous at $x \in B$ if for every sequence $\{x_n\}_{n=1}^{\infty}$ of elements from B which converges to x we have $f(x_n) \to f(x)$ (i.e. the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$). The function $f: B \to A$ is said to be continuous on B if it is continuous at every point $x \in B$.

The function $f: B \to A$ is said to be uniformly continuous on B if for each $\varepsilon > 0_A$ there is a $\delta > 0_A$ such that $|f(x) - f(y)| < \varepsilon$ holds for all $x, y \in B$ with $|x - y| < \delta$.

Theorem 3. An Archimedean field $\langle A, +, ., < \rangle$ is complete if and only if for two arbitrary elements $a, b \in A$, $a < b$ the following assertion holds: Every function $f: [a, b] \to A$ which is continuous on $[a, b]$ is uniformly continuous on $[a, b]$.

Proof. 1. Let the Archimedean field $\langle A, +, ., < \rangle$ be complete. Let $a, b \in A$, $a < B$. Let us assume that there is such a function $f: [a, b] \to A$ continuous on $[a, b]$ that f is not uniformly continuous on $[a, b]$. Then there exists such an $\varepsilon_0 > 0_A$ that for each $\delta = 1/k$ ($k = 1, 2, \ldots$) there exist points $x_k, y_k \in [a, b]$ with

$$|x_k - y_k| < \frac{1}{k}, \tag{11}$$

$$|f(x_k) - f(y_k)| \geqq \varepsilon_0 \tag{11'}$$

The interval $[a, b]$ is a closed and bounded subset of A and therefore it is a compact set (see Lemma 2). Hence there exists such a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ which converges to an element $x_0 \in [a, b]$. It follows from (11) that $y_{n_k} \to x_0$. In view of continuity of f we have

$$|(x_{n_k}) - f(y_{n_k})| \to 0_A \qquad (k \to \infty).$$

and simultaneously on account of (11′) we have

$$|f(x_{n_k}) - f(y_{n_k})| \geqq \varepsilon_0 \qquad (k = 1, 2, \ldots).$$

This is a contradiction.

2. Let the Archimedean field $\langle A, +, ., < \rangle$ be not complete. Then the set A is not continuously ordered. Therefore there is such a set $H \neq 0$, $H \subset A$, that is bounded from above but H has not the least upper bound in A.

Choose $a, b \in A$ in such a way that a is not any upper bound for H and b is an upper bound for H. Define a function $f:[a, b] \to A$ in the following way: If $x \in [a, b]$ and there exists a point $y \in [a, b] \cap H$ with $x < y$, then we put $f(x) = 0_A$. In the countrary case we put $f(x) = 1_A$.

We shall show that f is a continuous function on $[a, b]$.

Let $z \in [a, b]$, $f(z) = 0_A$. Let

$$x_k \to z,\ x_k \in [a, b] \qquad (k = 1, 2, \ldots). \tag{12}$$

Since $f(z) = 0_A$, there exists $y \in [a, b] \cap H$ such that $z < y$. It follows from (12) that for all sufficiently large k (for $k > k_0$) we have $x_k < y$. Therefore $f(x_k) = 0_A$ for each $k > k_0$ and hence $f(x_k) \to f(z)$.

Let $v \in [a, b]$, $f(v) = 1_A$. Then $v \notin H$ and there is no $y \in [a, b] \cap H$ with $y \geqq v$. Therefore for each $y \in [a, b] \cap H$ we have $y < v$. Since H has not the least upper bound, there exists $v' < v$ such that v' is an upper bound for H. Let $x_k \to v$. Then for all sufficiently large k (for $k > k_1$) we have $v' < x_k$ and therefore $f(x_k) = 1_A$ (for each $k > k_1$). Hence $f(x_k) \to f(v)$.

Thus f is a continuous function on $[a, b]$. We shall prove that the function f is not uniformly continuous on $[a, b]$.

Consider that $f(a) = 0_A$ and $f(b) = 1_A$. If $f((a + b)/2) = 0_A$, then we put $x_1 = (a + b)/2$, $y_1 = b$. If $f((a + b)/2) = 1_A$, then we put $x_1 = a$, $y_1 = (a + b)/2$. In both cases we have $y_1 - x_1 = (b - a)/2$. In such a way (by induction) we construct two sequences of elements of A

$$x_1 \leqq x_2 \leqq \ldots \leqq x_k \leqq \ldots,$$

$$y_1 \geqq y_2 \geqq \ldots \geqq y_k \geqq \ldots$$

such that

$$y_k - x_k = \frac{y_1 - x_1}{2^{k-1}} \to 0_A \qquad (k \to \infty), \tag{13}$$

further $f(x_k)=0_A$, $f(y_k)=1_A$ $(k=1, 2, \ldots)$.

Let $\delta > 0_A$. Then for suitable m on account of (13) we have $|x_m - y_m| < \delta$ and simultaneously for this m we have

$$|f(x_m) - f(y_m)| = 1_A > 0_A .$$

Thus the function f is not uniformly continuous on $[a, b]$. This ends the proof.

Definition 2. Let $\langle A, +, ., < \rangle$ be an ordered field. A function f: $A \to A$ is said to have the Darboux property if from the condition $f(x_1) < z < f(x_2)$ $(x_1, x_2, z \in A)$ the existence of such an $x \in A$ follows that x belongs to the open interval with the endpoints x_1, x_2 and $f(x) = z$.

Theorem 4. An Archimedean ordered field $\langle A, +, ., < \rangle$ is complete if and only if every function f: $A \to A$ which is continuous on A, has the Darboux property.

Proof. 1. Let the Archimedean field $\langle A, +, ., < \rangle$ be complete. Then the set $(A, <)$ is continously ordered. Let f: $A \to A$ be a continuous function on A and let

$$f(x_1) < z < f(x_2),$$

where x_1, x_2, $z \in A$. Let e.g. $x_1 < x_2$. Put

$$H = \{x \in [x_1, x_2] ; f(x) < z\}$$

Then $H \neq \emptyset$ for $x_1 \in H$. Further the set H is bounded from above (by the element x_2). Therefore there exists $t = \sup H \in (x_1, x_2)$. We shall prove that $f(t) = z$.

If $f(t) < z$, then in view of continuity of f at t we see that there exists such an interval (c, d) containing t that for each $x \in (c, d)$ we have $f(x) < z$. But this contradicts the definition of the point t.

It follows from the continuity of the function f and the definition of t that $f(t) \leqq z$. Thus we have $f(t) = z$. Hence the function f has the Darboux property.

2. Let the Archimedean field $\langle A, +, ., < \rangle$ be not complete. Then the set $(A, <)$ is not continuously ordered. Therefore there is such a non-empty set B, $B \subset A$, which is bounded from above and has not the least upper bound in A. Choose $a \in A$ and $b \in A$ such that a is not any upper bound for B and b is an upper bound for B. Then $a < b$. Define f: $[a, b] \to A$ in the same way as in the proof of Theorem 3. Then as we have seen f is a continuous function on $[a, b]$ and $f(a) = 0_A$, $f(b) = 1_A$. Choose in the definition of the Darboux property (see Definition 2) $z = 1/2$. Then $f(a) < 1/2 < f(b)$, but there is no $x \in (a, b)$ with $f(x) = 1/2$ because the values of the function f are only 0_A and 1_A. The proof is finished.

REFERENCES

?1] Bell, H. E.: Proof of a fundamental theorem on sequences. Amer. Math. Monthly 71 (1974), 665—666.

[2] Cohen, L. W.—Ehrlich, G.: The Structure of the Real Number System. D. van Nostrand Comp. INC., Toronto—New York—London, 1963.

[3] Hewitt, E.—Stromberg, K.: Real and Abstract Analysis. Springer-Verlag, Berlin—Heidelberg—New York, 1969.

Author's address:

Received: 17. 6. 1981

Tibor Šalát
Katedra algebry a teórie čísel MFF UK
Mlynská dolina
842 15 Bratislava

Anna Tarabová
Gymnázium Kežmarok, Hviezdoslavova 20
060 01 Kežmarok

SÚHRN

O VLASTNOSTIACH ARCHIMEDOVSKY USPORIADANÝCH POLÍ, KTORÉ SÚ EKVIVALENTNÉ ÚPLNOSTI

Tibor Šalát, Anna Tarabová, Bratislava

V práci sú sformulované štyri vlastnosti archimedovsky usporiadaných polí, ktoré sú ekvivalentné úplnosti poľa. Dve z nich sú formulované pomocou podmnožín poľa A, ostatné pomocou vlastností funkcií typu $f: B \to A$, $B \subset A$.

О СВОЙСТВАХ АРХИМЕДОВЫХ ПОЛЕЙ КОТОРЫЕ ЭКВИВАЛЕНТНЫ ПОЛНОТЕ

Тибор Шалат, Анна Тарабова, Братислава

В работе сформулированы четыре свойства архимедовых полей которые эквивалентны полноте полей. Два из них сформулированы при помощи полмножеств поля A, остальные используя функций типа $f: B \to A$, $B \subset A$.