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Expository papers

Regular polyhedra–old and new

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Abstract. Although it is customary to define *polygons* as certain families of edges, when considering *polyhedra* it is usual to view polygons as 2-dimensional pieces of the plane. If this rather illogical point of view is replaced by consistently understanding polygons as 1-dimensional complexes, the theory of polyhedra becomes richer and more satisfactory. Even with the strictest definition of regularity this approach leads to 17 individual regular polyhedra in the Euclidean 3-space and 12 infinite families of such polyhedra, besides the traditional ones (which consist of 5 Platonic polyhedra, 4 Kepler–Poinsot polyhedra, 3 planar tessellations and 3 Petrie–Coxeter polyhedra). Among the many still open problems that naturally arise from the new point of view, the most obvious one is the question whether the regular polyhedra found in the paper are the only ones possible in the Euclidean 3-space.

1. Introduction

Regular polyhedra have been investigated since antiquity. With the passage of time there have been many changes in points of view about them, and even in the definitions of the notions of polyhedra and of regularity. No formal consensus appears to have been reached so far, and virtually every condition that is imposed in some definition proposed in the literature is omitted or even contradicted in another – equally reasonable – definition. While the effects of the differences in definitions are rather superficial in respect to convex polyhedra, they have far-reaching consequences as soon as non-convex polyhedra are considered.

The present paper grew out of an attempt to provide definitions which would be natural, simple and elegant, while at the same time allowing interactions with classical geometry as well as with novel directions of research. Though all the ideas involved have appeared in the writings of various authors, it was the reading of Coxeter's beautiful "Regular Complex Polytopes" (Coxeter [1974]) that helped bring into coherent focus the relevant investigations that occupied me during the past year or so.

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The main aim of the following pages is the presentation of a new list, conjectured to be complete, of regular polyhedra in Euclidean 3-space E^3 (Section 3). Many of the extensions possible in different directions are briefly touched upon in the last Section, which also contains historical remarks and references. As a prerequisite for the main topic we discuss regular polygons in Section 2.

It is well known that the "traditional" regular polyhedra appear naturally in many contexts that have no apparent relation to regularity (see, for example, Fejes Tóth [1964]). Hence it is not surprising that many of the "new" regular polyhedra (or their 1-skeleta) have been found to possess remarkable properties even before they were recognized as "regular polyhedra." We shall enlarge on this aspect in Section 4.

Many helpful suggestions of a referee are gratefully acknowledged.

2. Regular polygons

Following Poinsot [1810] we shall define a *finite polygon* (or *n-gon*) $P = [V_1, V_2, \ldots, V_n]$ in a Euclidean space E^k as the figure formed by the distinct points (*vertices*) $V_1, \ldots, V_n$ of E^k, together with the segments (*edges*) $[V_i V_{i+1}]$ for $i = 1, 2, \ldots, n-1$, and $[V_n, V_1]$. An *infinite polygon* $P = [\ldots, V_{-1}, V_0, V_1, V_2, \ldots]$ consists of a sequence of distinct points (*vertices*) V_i, and of segments (*edges*) $[V_i, V_{i+1}]$, $i = 0, \pm 1, \pm 2, \ldots$, such that each compact subset of E^k meets only finitely many edges. Each edge is said to be incident with each of the two vertices that are its endpoints. If P is a polygon, a *flag* of P is a pair consisting of a vertex of P and an edge of P that is incident with that vertex.

A polygon P is said to be *regular* privided the group of its *symmetries* (that is, of the isometric homeomorphisms of E^k onto itself that map P onto itself) acts transitively on the family of all flags of P.

A systematic discussion of regular polygons may be found in Chapter 1 of Coxeter [1974]. In order to make the present paper selfcontained, and also in order to introduce convenient notation, we shall briefly review the relevant facts.

It is useful to classify the regular polygons into seven groups. As the polygons are rather well known we shall refrain from detailed descriptions, referring the reader to Figures 1, 2, and 3 instead.

Group 1. *Convex n-gon.* Symbol $\{n\}$, defined for each $n \geq 3$. (See Figure 1a.)

Group 2. *Star n-gon of density d.* Symbol $\{n/d\}$, defined whenever $1 < d < n/2$ with n and d coprime. (See Figure 1b.) (We could allow $d = 1$ in the above definition, obtaining $\{n\} = \{n/1\}$.)

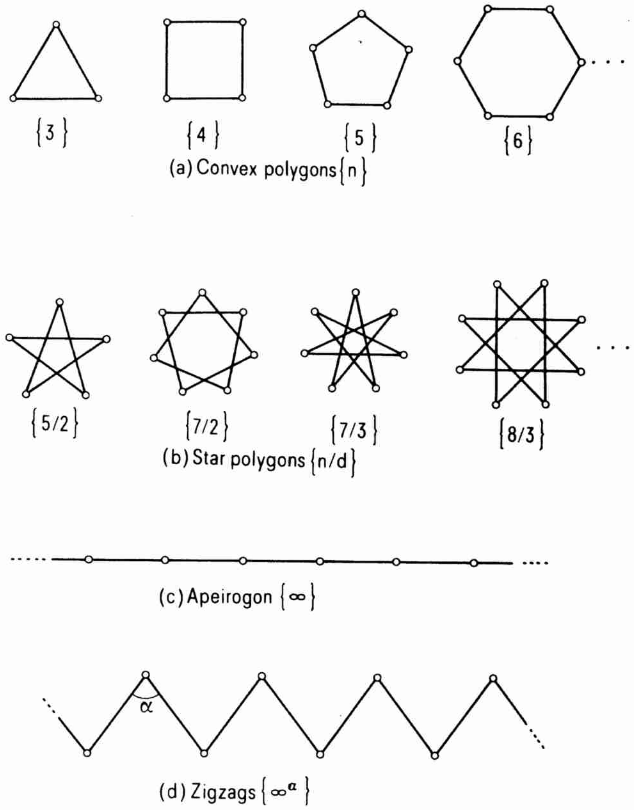

Figure 1

Group 3. *Apeirogon.* A single polygon, with symbol $\{\infty\}$. (See Figure 1c.)

Group 4. *Zigzag with angle* α. Symbol $\{\infty^\alpha\}$, defined for each α with $0<\alpha<\pi$. (See Figure 1d.) Note that $\{\infty\}$ could be interpreted as $\{\infty^\pi\}$.

Group 5. *Antiprismatic n-gon.* Symbol $\{n^\alpha/d\}$, where n is even, $1\le d<n/2$, n and d are coprime, and $0<\alpha<(n-2d)\pi/n$. (See Figure 2a.) The vertices of $\{n^\alpha/d\}$ may be obtained from those of $\{n\}$ by alternately raising and lowering them perpendicularly to the plane of $\{n\}$. When α tends to the upper bound, the polygons $\{n^\alpha/d\}$ tend to $\{n/d\}$.

Group 6. *Prismatic n-gon.* Symbol $\{2\cdot k^\alpha/d\}$, where $n=2k$ is even, k and $2d$ are coprime, $1\le d<k/2$, and $0<\alpha<(k-2d)\pi/k$. (See Figure 2b.) (The vertices coincide with those of a right prism based on $\{k\}$.)

Group 7. *Helical polygon.* Symbol $\{\infty^{\alpha,\beta}\}$, where $0<\alpha<\pi$, $0<\beta<\pi$, $\alpha+\beta>\pi$. Vertices V_j, $j=0,\pm1,\pm2,\dots$ lie on the helix given parametrically by $(a\cos\beta t, a\sin\beta t, bt)$, where V_j results for $t=j$, and α is the angle between successive edges.

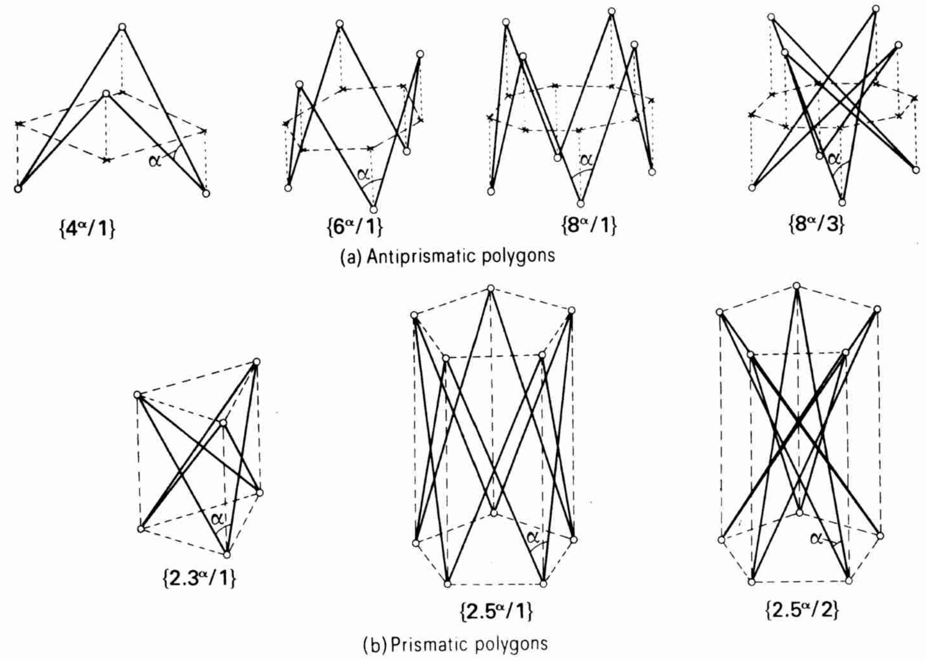

Figure 2

Note that if $ab > 0$ we have a *right* helix, while $ab < 0$ is a *left* helix. Hence each $\{\infty^{\alpha,\beta}\}$ comes in two enantiomorphic forms, left and right. See Figure 3.

Rather obviously, polygons in groups 1, 2, 3, 4 are planar, those in groups 5, 6, 7 are skew. Polygons in groups 1, 2, 5, 6 are finite, the others infinite. It should be noted that only for polygons in group 1 is a part of the plane naturally associated with the polygon; for other types no such association is possible, although in some cases parts of certain 2-dimensional manifolds may be associated with the polygon. We shall discuss this in detail in Section 4.

3. Regular polyhedra

A *polyhedron* P is any family of polygons (called *faces* of P) that has the following properties:

(i) Each edge of one of the faces is an edge of just one other face.

(ii) The family of polygons is connected; that is, for any two edges E and E' of P there exists a chain $E = E_0, P_1, E_1, P_2, E_2, \ldots, P_n, E_n = E'$ of edges and faces of P, where each P_i is incident with E_{i-1} and with E_i.

(iii) Each compact set meets only finitely many faces.

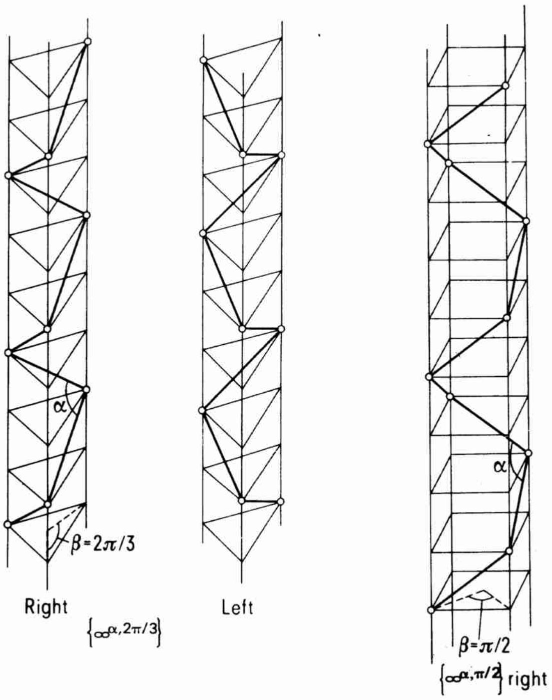

Figure 3
Helical polygons.

A polyhedron is said to be incident with each of its faces, as well as with each edge and each vertex of each of its faces.

A *flag* of a polyhedron P is any triple consisting of a vertex, an edge, and a face of P, all mutually incident. The polyhedron P is *regular* if its group of symmetries acts transitively on the family of its flags.

Let P be a polyhedron, V one of its vertices, and $[V, V_i]$, $i = 1, 2, \ldots, k$ all the edges of P incident with V. The *vertex figure* P/V of P at V consists of the points V_i, $i = 1, 2, \ldots, k$ as vertices, and of those segments $[V_i, V_j]$ as edges for which $[V, V_i]$ and $[V, V_j]$ are edges of P incident with one of the faces of P. Therefore the vertex figure of P at V consists of one or more polygons. For a regular polyhedron P the vertex figure P/V is clearly independent of the vertex V. For each regular polyhedron in E^3 known to us, the vertex figure happens to be a single polygon which is, naturally, regular (see also Remark (7) below).

Following the convenient custom introduced by Schläfli and elaborated in particular by Coxeter, we shall assign to each type of regular polyhedra in E^3 a *Schläfli symbol* $\{A, B\}$, where $\{A\}$ is the symbol for the regular polygons that are faces of P, while $\{B\}$ is the symbol for the vertex figures of the polyhedron P.

In the description of many of the "new" regular polyhedra we find it useful to consider the so-called Petrie polygons (see, for example, Coxeter [1973, p. 24]). For a polyhedron P, a *Petrie polygon* $\prod$ of P is a polygon with vertices among the vertices of P, and edges chosen among the edges of P so that each two successive edges of $\prod$ are incident with one face of P, but no three successive edges of $\prod$ are incident with the same face of P. It is easy to see that for each regular polyhedron P the family of all Petrie polygons of P is a regular polyhedron $\pi(P)$, the *Petrie polyhedron* of P. Moreover, $\pi(\pi(P))$ coincides with P.

We find it convenient to group the regular polyhedra possible in Euclidean 3-space E^3 into 8 classes, of which the first four are well known.

Class 1. *Platonic polyhedra* are the 5 finite regular polyhedra in which faces as well as vertex figures are convex polygons. It is well known that the Platonic polyhedra are:

$\{3, 3\}$ tetrahedron;
$\{3, 4\}$ octahedron;
$\{4, 3\}$ hexahedron, or cube;
$\{3, 5\}$ icosahedron;
$\{5, 3\}$ dodecahedron.

They are illustrated in Figure 4.

Class 2. *Planar tessellations* are the infinite regular polyhedra in which faces as well as vertex figures are convex polygons. The three planar tessellations are $\{4, 4\}$, $\{3, 6\}$, and $\{6, 3\}$; they are shown in Figure 5.

Class 3. The *Kepler–Poinsot polyhedra* are the finite regular polyhedra in which the faces are convex polygons and the vertex figures star polygons, or the

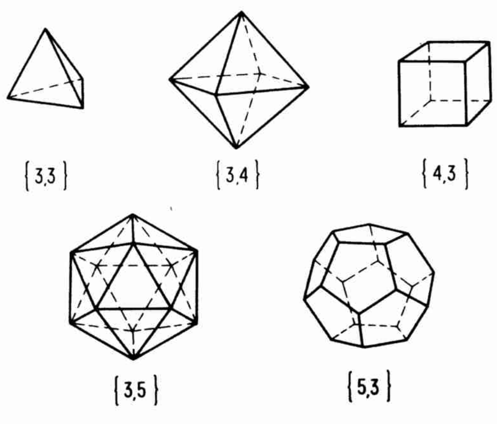

Figure 4
Platonic polyhedra.

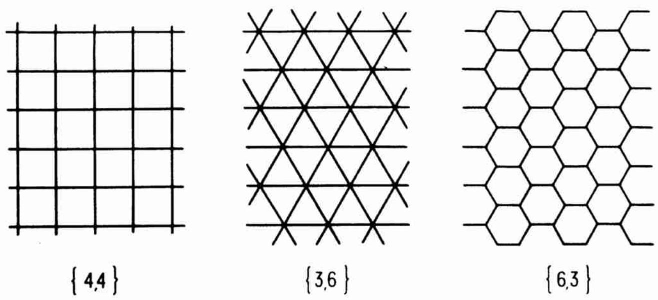

Figure 5
Planar tessellations.

other way around. The four Kepler–Poinsot polyhedra are:

{5, 5/2} great dodecahedron;
{3, 5/2} great icosahedron;
{5/2, 5} small stellated dodecahedron;
{5/2, 3} great stellated dodecahedron.

They are illustrated in Figure 6.

Class 4. The *Petrie–Coxeter polyhedra* are the infinite regular polyhedra with convex polygons as faces, and antiprismatic polygons as vertex figures. The three

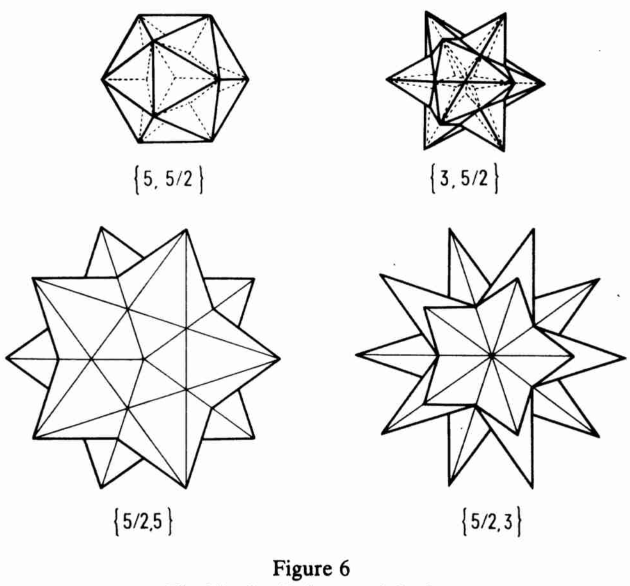

Figure 6
The Kepler-Poinsot polyhedra.

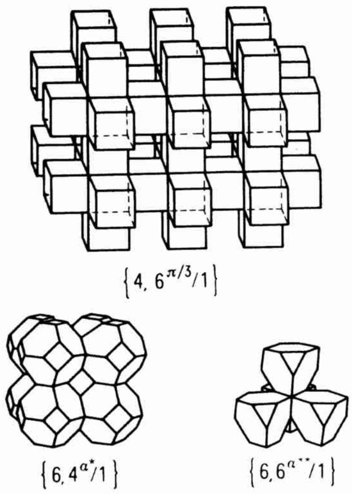

Figure 7
Fragments of the Petrie–Coxeter polyhedra.

Petrie–Coxeter polyhedra are $\{4, 6^{\pi/3}/1\}$, $\{6, 4^{\alpha^*}/1\}$ (where $\alpha^* = \arccos 2/3 \doteq 48°12'$), and $\{6, 6^{\alpha^{**}}/1\}$ (where $\alpha^{**} = \arccos 5/6 \doteq 33°33'$); they are illustrated in Figure 7.

Class 5. *Finite regular polyhedra* with *finite skew polygons* as faces. There are nine polyhedra in this class; they are listed and described in Table 1, and illustrated in part in Figure 8. The completeness of the list may be established by exhaustive checking, since the vertices of each finite regular polyhedron must coincide with those of a Platonic polyhedron. Each polyhedron $\{n^{\alpha}/d, q\}$ of class 5 is the Petrie polyhedron of a Platonic or Kepler–Poinsot polyhedron $\{p, q\}$, where $\alpha = (p-2)\pi/p$ and $\cos^2(\pi/p) + \cos^2(\pi/q) = \cos^2(\pi/h)$, with $h = n/d$.

Table 1
The nine regular polyhedra of class 5.

Schläfli symbol	Number of vertices	Number of edges	Number of polygons	Relation to other regular polyhedra
$\{4^{\pi/3}/1, 3\}$	4	6	3	$\pi\{3, 3\}$
$\{6^{\pi/3}/1, 4\}$	6	12	4	$\pi\{3, 4\}$
$\{6^{\pi/2}/1, 3\}$	8	12	4	$\pi\{4, 3\}$
$\{10^{\pi/3}/1, 5\}$	12	30	6	$\pi\{3, 5\}$
$\{6^{\pi/5}/1, 5\}$	12	30	10	$\pi\{5/2, 5\}$
$\{6^{3\pi/5}/1, 5/2\}$	12	30	10	$\pi\{5, 5/2\}$
$\{10^{\pi/3}/3, 5/2\}$	12	30	6	$\pi\{3, 5/2\}$
$\{10^{3\pi/5}/1, 3\}$	20	30	6	$\pi\{5, 3\}$
$\{10^{\pi/5}/3, 3\}$	20	30	6	$\pi\{5/2, 3\}$

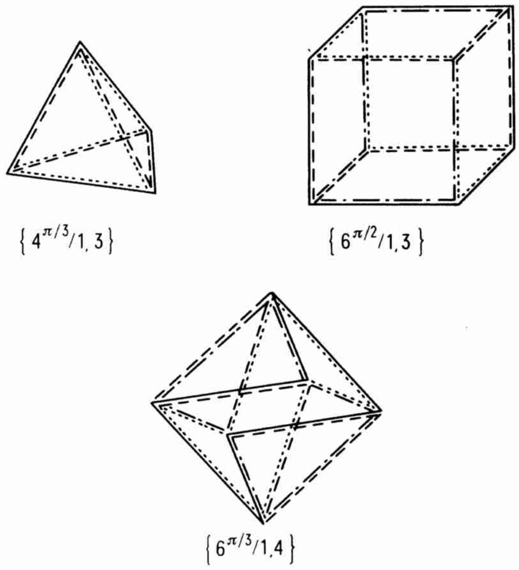

Figure 8
Finite regular polyhedra with finite skew polygons.

Class 6. *Infinite regular polyhedra* with *finite skew polygons* as faces. Three infinite families of such polyhedra are known, and also three polyhedra not in those families (see Table 2; compare also remark (3) below). The three infinite families are related to the planar tessellations.

$\{4^{\alpha}/1, 4\}$ for $0<\alpha<\pi/2$ is obtained on deforming the squares of $\{4, 4\}$ into skew polygons $\{4^{\alpha}/1\}$ by perpendicularly lifting alternate vertices of $\{4, 4\}$ to equal heights above the plane of $\{4, 4\}$. For $\alpha \to \pi/2$ the polyhedra tend to $\{4, 4\}$.

$\{6^{\alpha}/1, 3\}$ for $0<\alpha<2\pi/3$ is similarly obtained by deforming the hexagons of $\{6, 3\}$ into skew hexagons $\{6^{\alpha}/1\}$; for $\alpha \to 2\pi/3$ the polyhedra tend to $\{6, 3\}$.

$\{2.3^{\alpha}/1, 6\}$ for $0<\alpha<\pi/3$ is obtained from two parallel copies of $\{3, 6\}$, one above the other, on replacing pairs of triangles by skew hexagons $\{2.3^{\alpha}/1\}$ having the same vertices. For $\alpha \to \pi/3$ the polyhedra tend to the doubly covered $\{3, 6\}$.

The remaining three polyhedra known in class 6 are $\{6^{\pi/3}/1, 6\}$, $\{4^{\pi/3}/1, 6\}$ and $\{6^{\pi/2}/1, 4\}$. The first consists of the vertex figures of alternate vertices of the Petrie–Coxeter polyhedron $\{4, 6^{\pi/3}/1\}$; the second is the Petrie polyhedron of the first, and may alternatively be described as formed by one skew quadrangle $\{4^{\pi/3}/1\}$ inscribed into each of three-fourths of the cubes of the tessellation $\{4, 3, 4\}$. The third may be obtained by taking one Petrie polygon in each of one half of the cubes that form the tessellation $\{4, 3, 4\}$ (see Figure 9); it is self-Petrie in the sense that its Petrie polyhedron is again $\{6^{\pi/2}/1, 4\}$.

Class 7. *Regular polyhedra with zigzag polygons.* This class is known to contain six infinite families of polyhedra. The first three families are clearly related to the plane tessellations.

$\{\infty^\alpha, 4\}$, for $0<\alpha\le\pi/2$, is $\pi\{4^\alpha/1, 4\}$ unless $\alpha=\pi/2$, in which case it is $\pi\{4, 4\}$.

$\{\infty^\alpha, 3\}$, for $0<\alpha\le 2\pi/3$, is $\pi\{6^\alpha/1, 3\}$ unless $\alpha=2\pi/3$, in which case it is $\pi\{6, 3\}$.

$\{\infty^\alpha, 6\}$, for $0<\alpha\le\pi/3$, is $\pi\{2.3^\alpha/1, 6\}$ unless $\alpha=\pi/3$, in which case it is $\pi\{3, 6\}$.

The members of the other three families are Petrie polyhedra of regular polyhedra of class 8; in contrast to the ones just described, they are not contained in any slab of finite width.

$$\{\infty^{\alpha(b)}, 4^{\alpha^*(b)}/1\}, \quad \text{where} \quad \alpha(b)=\arccos -b^2/(b^2+1),$$

$$\alpha^*(b)=\arccos 2b^2/(2b^2+1),\ b\ne 0, \text{ is } \pi\{\infty^{\alpha(b),\pi/2}, 4^{\alpha^*(b)}/1\}.$$

$$\{\infty^{\gamma(b)}, 6^{\gamma^*(b)}/1\}, \quad \text{where} \quad \gamma(b)=\arccos (1-2b^2)/(2+2b^2),$$

$$\gamma^*(b)=\arccos (8b^2-1)/(8b^2+2),\ b\ne 0, \text{ is } \pi\{\infty^{\gamma(b),2\pi/3}, 6^{\gamma^*(b)}/1\}.$$

$$\{\infty^{\delta(b)}, 2.3^{\delta^*(b)}/1\}, \quad \text{where} \quad \delta(b)=\arccos -(1+2b^2)/(2+2b^2),$$

$$\delta^*(b)=\arccos (8b^2+3)/(8b^2+6),\ b\ne 0, \text{ is } \pi\{\infty^{\delta(b),\pi/3}, 2.3^{\delta^*(b)}/1\}.$$

Table 2
Regular polyhedra of class 6.

Schläfli symbol	Description	Relation to other regular polyhedra
$\{4^\alpha/1, 4\}$ for $0<\alpha<\pi/2$	Deformation of $\{4, 4\}$	$\pi\{\infty^\alpha, 4\}$
$\{6^\alpha/1, 3\}$ for $0<\alpha<2\pi/3$	Deformation of $\{6, 3\}$	$\pi\{\infty^\alpha, 3\}$
$\{2.3^\alpha/1, 6\}$ for $0<\alpha<\pi/3$	Obtainable from $\{3, 6\}$	$\pi\{\infty^\alpha, 6\}$
$\{6^{\pi/3}/1, 6\}$	One-half of the vertex figures of $\{4, 6^{\pi/3}/1\}$	$\pi\{4^{\pi/3}/1, 6\}$
$\{4^{\pi/3}/1, 6\}$	One quadrangle $\{4^{\pi/3}/1\}$ in each of 3/4 of the cubes of $\{4, 3, 4\}$	$\pi\{6^{\pi/3}/1, 6\}$
$\{6^{\pi/2}/1, 4\}$		

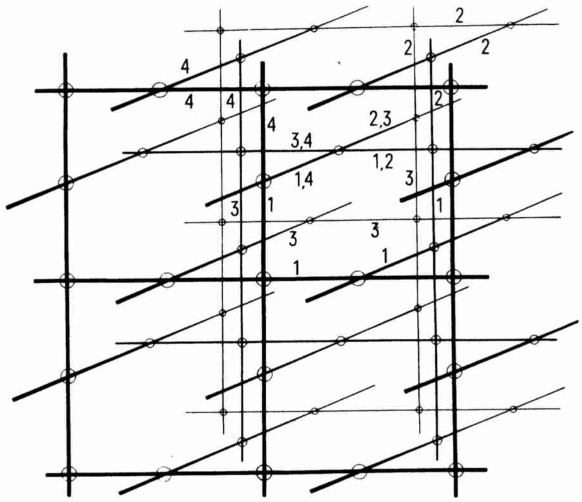

Figure 9
A fragment of the regular polyhedron $\{6^{\pi/2}/1, 4\}$. Each of the numerals 1, 2, 3, 4, indicates the edges of one of the hexagons $\{6^{\pi/2}/1\}$.

In particular, if $b = 1/\sqrt{2}$ then $\gamma(b) = \pi/2$ and $\gamma^*(b) = \pi/3$; the edges of the resulting polyhedron $\{\infty^{\pi/2}, 6^{\pi/3}/1\}$ coincide with those of the Petrie–Coxeter polyhedron $\{4, 6^{\pi/3}/1\}$ (or of the cubic lattice). The polyhedron is illustrated in Figure 10.

Class 8. *Regular polyhedra with helical polygons.* Five such polyhedra are known, besides three infinite families.

In $\{\infty^{2\pi/3,\pi/2}, 3\}$ the helices follow, in each direction, one quarter of the stacks of cubes in the cubic tessellation $\{4, 3, 4\}$; see Figure 11.

$\{\infty^{2\pi/3,2\pi/3}, 3\}$ is $\pi\{\infty^{2\pi/3,\pi/2}, 3\}$.

Each of those two types exists in two enantiomorphic forms, using only left or only right helices.

$\{\infty^{\pi/2,2\pi/3}, 6^{\pi/3}/1\}$ is $\pi\{4, 6^{\pi/3}/1\}$.

$\{\infty^{2\pi/3,2\pi/3}, 4^{\pi/3}/1\}$ is $\pi\{6, 4^{\alpha^*}/1\}$ where $\alpha^* = \arccos 2/3 \doteq 48°12'$.

$\{\infty^{2\pi/3,\pi/2}, 6^{\alpha^{**}}/1\}$ is $\pi\{6, 6^{\alpha^{**}}/1\}$, where $\alpha^{**} = \arccos 5/6 \doteq 33°33'$.

The three infinite families consist of helical polygons "rising" above the polygons in planar tessellations. To simplify the notation, in the helical polygons

Table 3
Regular polyhedra of class 7.

Schläfli symbol	Description	Relation to other regular polyhedra
$\{\infty^{\alpha}, 4\}$ for $0<\alpha\leq\pi/2$	Deformation of zigzags in the 1-skeleton of $\{4, 4\}$	$\pi\{4^{\alpha}/1, 4\}$ for $0<\alpha<\pi/2$ $\pi\{4, 4\}$ for $\alpha=\pi/2$
$\{\infty^{\alpha}, 3\}$ for $0<\alpha\leq 2\pi/3$	Deformation of zigzags in the 1-skeleton of $\{6, 3\}$	$\pi\{6^{\alpha}/1, 3\}$ for $0<\alpha<2\pi/3$ $\pi\{6, 3\}$ for $\alpha=2\pi/3$
$\{\infty^{\alpha}, 6\}$ for $0<\alpha\leq\pi/3$	Deformation of zigzags in the 1-skeleton of $\{3, 6\}$	$\pi\{2.3^{\alpha}/1, 6\}$ for $0<\alpha<\pi/3$ $\pi\{3, 6\}$ for $\alpha=\pi/3$
$\{\infty^{\alpha(b)}, 4^{\alpha^*(b)}/1\}$, $b\neq 0$		$\pi\{\infty^{\alpha(b),\pi/2}, 4^{\alpha^*(b)}/1\}$
$\{\infty^{\gamma(b)}, 6^{\gamma^*(b)}/1\}$, $b\neq 0$		$\pi\{\infty^{\gamma(b),2\pi/3}, 6^{\gamma^*(b)}/1\}$
$\{\infty^{\delta(b)}, 2.3^{\delta^*(b)}/1\}$, $b\neq 0$		$\pi\{\infty^{\delta(b),\pi/3}, 2.3^{\delta^*(b)}/1\}$

For the definition of the functions $\alpha(b)$, $\alpha^*(b)$, etc. see the text.

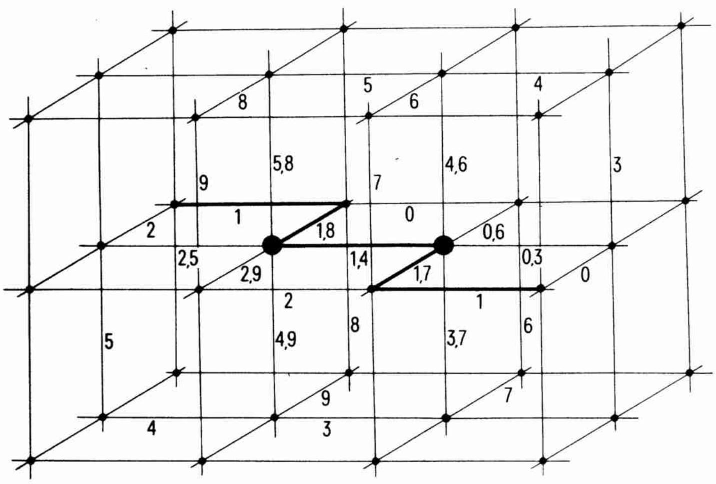

Figure 10
A fragment of the regular polyhedron $\{\infty^{\pi/2}, 6^{\pi/3}/1\}$. Edges belonging to the polygons incident with at least one of the two emphasized vertices are indicated by numerals from 0 to 9. Polygon #1 is indicated by heavy edges.

Table 4
Regular polyhedra of class 8.

Schläfli symbol		Relation to other regular polyhedra
$\{\infty^{2\pi/3,\pi/2}, 3\}$		$\pi\{\infty^{2\pi/3,2\pi/3}, 3\}$
$\{\infty^{2\pi/3,2\pi/3}, 3\}$		$\pi\{\infty^{2\pi/3,\pi/2}, 3\}$
$\{\infty^{\pi/2,2\pi/3}, 6^{\pi/3}/1\}$		$\pi\{4, 6^{\pi/3}/1\}$
$\{\infty^{2\pi/3,2\pi/3}, 4^{\pi/3}/1\}$		$\pi\{6, 4^{\alpha^*}/1\}$
$\{\infty^{2\pi/3,\pi/2}, 6^{\alpha^{**}}/1\}$		$\pi\{6, 6^{\alpha^{**}}/1\}$
$\{\infty^{\alpha(b),\pi/2}, 4^{\alpha^*(b)}/1\}$,	$b\neq 0$	$\pi\{\infty^{\alpha(b)}, 4^{\alpha^*(b)}/1\}$
$\{\infty^{\gamma(b),2\pi/3}, 6^{\gamma^*(b)}/1\}$,	$b\neq 0$	$\pi\{\infty^{\gamma(b)}, 6^{\gamma^*(b)}/1\}$
$\{\infty^{\delta(b),\pi/3}, 2.3^{\delta^*(b)}/1\}$,	$b\neq 0$	$\pi\{\infty^{\delta(b)}, 2.3^{\delta^*(b)}/1\}$

For the definitions of α^*, α^{**}, $\alpha(b)$, etc. see the text.

used we have assumed that $a=1/\sqrt{2}$ in the first family, $a=1/\sqrt{3}$ in the second, and $a=1$ in the third, while $b\neq 0$ is freely variable. The three families are:

$$\{\infty^{\alpha(b),\pi/2}, 4^{\alpha^*(b)}/1\}, \quad \text{where} \quad \alpha(b)=\arccos -b^2/(b^2+1) \quad \text{and} \quad \alpha^*(b)=\arccos 2b^2/(2b^2+1).$$

$$\{\infty^{\gamma(b),2\pi/3}, 6^{\gamma^*(b)}/1\}, \quad \text{where} \quad \gamma(b)=\arccos (1-2b^2)/(2+2b^2), \quad \text{and} \quad \gamma^*(b)=\arccos (8b^2-1)/(8b^2+2).$$

$$\{\infty^{\delta(b),\pi/3}, 2.3^{\delta^*(b)}/1\}, \quad \text{where} \quad \delta(b)=\arccos -(1+2b^2)/(2+2b^2) \quad \text{and} \quad \delta^*(b)=\arccos (8b^2+3)/(8b^2+6).$$

It should be noted that the polyhedron obtained in the second of those families for $b=1/\sqrt{2}$ (so that $\gamma(b)=\pi/2$, $\gamma^*(b)=\pi/3$), which we could denote $\{\infty^{\pi/2,2\pi/3}, 6^{\pi/3}/1\}^*$, is distinct from the polyhedron with the *same* Schläfli symbol mentioned earlier. One is $\pi\{\infty^{\pi/2}, 6^{\pi/3}/1\}$, while the other is $\pi\{4, 6^{\pi/3}/1\}$. This phenomenon is possible because of the existence of enantiomorphic forms of the helical polygons $\{\infty^{\pi/2,2\pi/3}\}$. In $\pi\{4, 6^{\pi/3}/1\}$ two polygons have at most one edge in common, while in $\pi\{\infty^{\pi/2}, 6^{\pi/3}/1\}$ polygons with a common edge have infinitely many common edges.

This completes the list of the regular polyhedra in Euclidean 3-space known at the present. It may be conjectured that the list is final, and that no other such polyhedra exist.

4. Remarks

(1) The regular polygons (in Euclidean 3-space) – finite and infinite, planar and skew – were explicitly considered by Coxeter [1937], although attention was

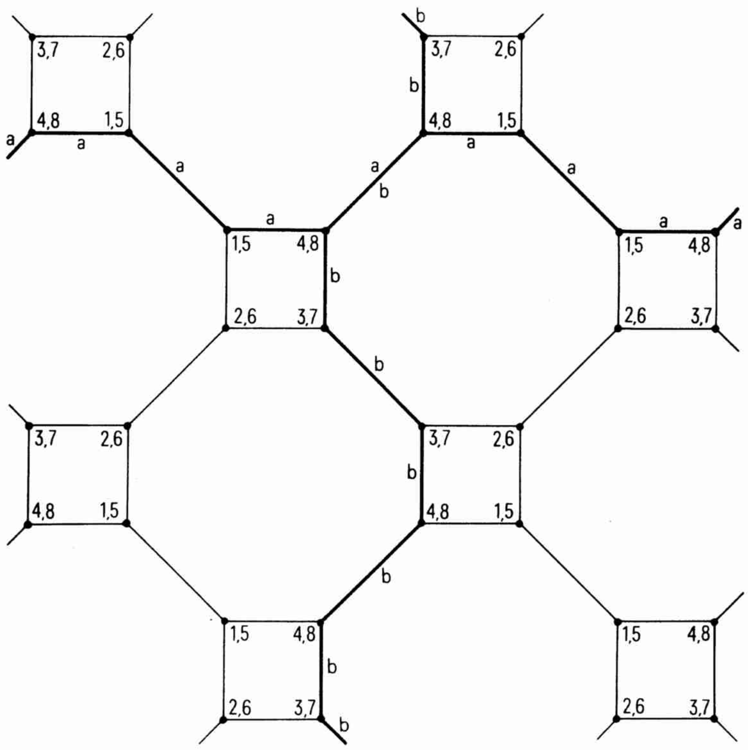

Figure 11
An orthogonal projection of a fragment of the regular polyhedron $\{\infty^{2\pi/3,\pi/2}, 3\}$. Two turns of vertical helices are represented by the squares (the numerals indicating successive vertices on the helices, equal numerals corresponding to the same height in all helices). Projections of edges of two horizontal helices are indicated by heavy lines and marked a (passing through vertices marked 4 and 5) and b (vertices 3 and 4).

restricted to simple ones. Finite, but not necessarily simple regular polygons in Euclidean spaces of all dimensions were investigated by Efremovič–Ilyašenko [1962]. A systematic treatment of the general case may be found in Coxeter [1974]. Our presentation in Section 2 differs in some aspects of the notation from Coxeter's.

As pointed out by a referee, in order to facilitate the extension to higher dimensions a good case could be made for the following changes in our notation: Instead of $\{n^\alpha/d\}$ for polygons in group 5 write $\{2 \cdot n^\alpha/d\}$, just as in group 6. A polygon $\{2 \cdot n^\alpha/d\}$ belongs to group 5 if n is even, to group 6 if n is odd; in either case the number of its vertices is the least common multiple of 2 and n (hence it is n in the former case, $2n$ in the latter), and in both cases the projections on the vertical line and horizontal plane are $\{2\}$ and $\{n/d\}$, respectively. Thus the groups 5 and 6 are actually one group. Similarly – and motivated by a similar concern for projections – a helical polygon $\{\infty^{\alpha,\beta}\}$ could be denoted by $\{k \cdot \infty^{\alpha}\}$, where $k = 2\pi/\beta$; for polygons in group 4 this would suggest the notation $\{2 \cdot \infty^{\alpha}\}$, since for them $\beta = \pi$ and so $k = 2$. However, since the real motivation and advantages of this notation become apparent only in higher dimensions, and since in the case of 3-dimensional space there are also several reasons for preferring the notation we used, the notation adopted has been retained.

It would appear to be of interest to investigate and completely classify some types of polygons slightly less symmetric than the regular ones. For example, it should be easy to determine all those for which the symmetries act transitively on the vertices, or the ones for which the symmetries act transitively on the edges.

(2) It is rather strange that although the perception of polygons as piecewise linear immersions of the 1-sphere goes back at least to Poinsot [1810] and is quite generally accepted, in the definition of polyhedra practically all authors use – either explicitly or implicity – polygons as pieces of the plane; see, for example, Brückner [1900, p. 2], Fejes Tóth [1965, pp. 101–102], Coxeter [1974, p. 12–13]. It is this "horror vacui" that is probably responsible for the fact that even the regular polyhedra in 3-space have not yet been completely investigated, and also for the following phenomenon: In the publications that deal with regular (or otherwise symmetric) polyhedra composed of skew regular polygons, the authors find it necessary to "span" the skew polyhedra by "membranes" that are chosen as minimal surfaces, or put together from planar polygons or from pieces of parabolic hyperboloids (see, for example, Burt [1966], Schoen [1968a], Williams [1972]).

(3) Some of the "new" regular polyhedra of classes 5 and 6 (or objects related to them) have been discussed to a certain extent earlier, at least with the interpretation of "skew polygon" as a part of some 2-manifold (see remark (2) above). Though there probably exist other publications dealing with this topic, those I am aware of are the following: Burt [1966] discusses (essentially) $\{4^{\alpha}/1, 4\}$ and $\{6^{\alpha}/1, 3\}$; the same "saddle tessellations" appear in Pearçe [1966] and Williams [1972]. Schoen [1968a] describes the three polyhedra of class 6, together with three other polyhedra; I do not understand the construction of those last three, but they seem not to be regular in our sense (lacking symmetries which interchange the two vertices of an edge of a polygon). Schoen [1968b] describes the nine members of class 5, and also $\{2.3^{\alpha}/1, 6\}$. For regular maps on 2-manifolds the notion analogous to that of "Petrie polyhedron" is well known; see, for example, Coxeter–Moser [1972, p. 112].

(4) In the case of the "traditional" regular polyhedra it is well known that their 1-skeleta (that is, the one-dimensional complexes formed by their vertices

and edges) are of interest in many other contexts. It appears that the situation is somewhat analogous with respect to several of the "new" polyhedra. For example, the 1-skeleton of the self-Petrie polyhedron $\{6^{\pi/2}/1, 4\}$ is the "net" #4 in Wells [1954b; Table 2 and Figure 6]; similarly, the 1-skeleton of $\{\infty^{\alpha_0}, 4^{\pi/3}/1\}$, where $\alpha_0 = \arccos(-1/3) \doteq 109°28' \cdots$ (as well as the 1-skeleton of its Petrie polyhedron $\{\infty^{\alpha_0, \pi/2}, 4^{\pi/3}/1\}$) is the well known "diamond net" (see, for example, net #1 in Wells [1954b; Table 2 and Figure 6]). The 1-skeleton of $\{\infty^{2\pi/3, \pi/2}, 3\}$ (and of its Petrie polyhedron $\{\infty^{2\pi/3, 2\pi/3}, 3\}$) is the net #1 in Wells [1954a; Table 2 and Figure 6]. A more thorough review of the literature would surely reveal many more extremal or otherwise remarkable properties of the "new" regular polyhedra and their 1-skeleta.

(5) The definition of regularity using flags appears to have been first proposed by Du Val [1964, p. 63], and was adopted in Coxeter [1974]. In special instances similar notions were used earlier in axiomatic geometry and in the study of finite geometries. It appears to be, at the same time, elegant, usable, very restrictive, and applicable in rather varied circumstances.

(6) One of the curious aspects of the way of looking at regular polyhedra presented here is the following. In the "traditional" classes 1, 2, 3 (and, with some effort, class 4) there is a rather natural pairing of mutually reciprocal (polar, dual) polyhedra (with $\{3, 3\}$, $\{4, 4\}$, and $\{6, 6^{\alpha^{**}}/1\}$ each being selfreciprocal), but the collections of Petrie polygons do not form any "traditionally acknowledged" polyhedron. In the new interpretation, the Petrie paths form regular polyhedra in each case, but no reasonable definition of "reciprocal pairs" has been found so far.

(7) In most definitions of polyhedra, and of regular polyhedra, it is specifically required that all faces incident with a vertex form one "circuit" (i.e., that each vertex figure be a single polygon). We have not found it necessary to impose this condition since all the regular polyhedra found above satisfied it automatically. It would be of interest to see whether it in general follows from the other requirements of regularity (it probably is not a consequence of some of the weaker conditions often used to define regularity).

(8) Quite a few other classes of more or less symmetrical polyhedra are interesting and deserve a detailed investigation. For example, there exist non-regular *totally transitive* polyhedra P such that all faces and vertex figures of P are regular and the symmetries of P act transitively on its vertices, its edges, and its faces. One such family are the "cylindrical polyhedra" formed by suitable polygons $\{4^{\alpha}/1\}$ (see Figure 12); they were considered by Burt [1966]. Between the totally transitive polyhedra and the polyhedra we call regular are those called "regular" by many other authors; for them the symmetries are only assumed to permute cyclically the vertices of each face, as well as the faces at each vertex. It

seems that the three polyhedra described by Schoen [1968a] and mentioned above in remark (3) are "regular" in this sense.

A larger class that appears to be important from various points of view are the *Archimedean polyhedra*, which have regular polygons as faces, and symmetries that act transitively on their vertices. Even the subclass consisting of those infinite Archimedean polyhedra in which the faces are all of one convex type, has not been completely investigated. Wachman, Burt and Kleinmann [1974] list many infinite families as well as individual polyhedra; they found polyhedra of the following types (denoting by n^p any Archimedean polyhedron in which all faces are $\{n\}$, with p faces meeting at each vertex): $3^6, 3^7, 3^8, 3^9, 3^{10}, 3^{12}, 4^4, 4^5, 4^6, 6^4, 6^6$. Some of those polyhedra appear also in Wells [1969], and in Gott [1967]; the latter also describes a remarkable Archimedean polyhedron of type 5^5, not mentioned in the literature before or after. The most challenging open problem concerning Archimedean polyhedra of types n^p is the chasm between the experimental "fact" that there are no types besides those just listed, and the absence of any theoretical explanation for that "fact."

(9) The classification of regular polygons and polyhedra in other spaces

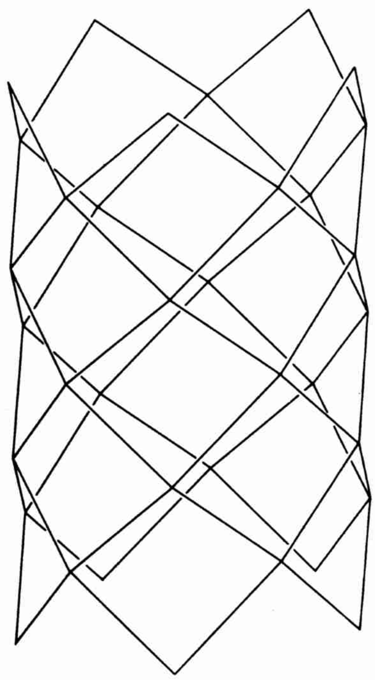

Figure 12
A fragment of a "cylindrical" polyhedron. (Adapted from Burt [1966].)

(Euclidean spaces of higher dimensions, hyperbolic, spherical and elliptic spaces), and in Euclidean (or other) spaces but with other groups allowed for symmetries (affine, projective, topological homeomorphisms, combinatorial isomorphisms) offer other fruitful directions of exploration. The first steps in some of those directions have already been made. For example, Coxeter [1937] investigated some of the regular "skew" polyhedra in higher-dimensional Euclidean spaces, while McMullen [1967], [1968] considered convex polyhedra (and polytopes) regular in the sense of affine, projective or combinatorial symmetries. The study of "regular maps" on 2-manifolds is too well known to require detailed references; see, for example, Coxeter–Moser [1972], Ringel [1974].

As with convex polygons in the plane, in some circumstances it is possible – and even desirable and useful – to associate with a finite polygon a 2-cell bounded by the polygon; a polyhedron may then give rise to a map on a manifold. In its turn, such a manifold may happen to divide a containing space in a suitable manner so that it becomes meaningful to speak of its interior, etc. Although often very convenient when naturally possible, the possibility of such "fleshing out" should not be required in the definition of regularity, or even of "polyhedron."

(10) There is no reason to stop at the stage of polyhedra the process of investigating regular figures. Connected families of (regular) polyhedra, in which each polygon is shared by two polyhedra and in which symmetries act transitively on flags (each composed of a vertex, an edge, a polygon, and a polyhedron, all mutually incident) could reasonably be called regular 3-topes. Similarly, regular d-topes may be defined for all larger d. As in the case of polyhedra, regularity may also be interpreted in non-metrical ways (combinatorial, topological, etc.). One example in this direction is provided by the forthcoming paper of Coxeter–Shephard [1977], in which (combinatorially) regular maps on the torus are used to put together a regular 3-tope (which happens to be a 3-sphere if the tori are interpreted as solid tori). Other such examples are given in Grünbaum [1977].

(11) The spirit of the present paper is probably best described by the desire to rid the theory of regular polyhedra of the psychologically motivated crutch of "membranes" spanning the polygons used as building blocks. The plethora of "new" regular polyhedra, and of naturally arising questions, would appear to lend justification to this way of thinking about polyhedra. However, one more vestige of the "anthropocentric" viewpoint still remains, and should – at least in some investigations – be removed: Instead of taking an edge to be a segment of a straight line, we could without any loss of precision (and with quite a gain in clarity in case of star polygons, for example) define an edge as an unordered pair of points (vertices). While this does not affect in any way the above considerations, it at once leads to the rather obvious possibility of considering "3-point edges," "4-point edges," etc., and using them to form "polygons," "polyhedra," etc. This

generalization – which can be geared to a combinatorial, topological, or metric point of view – brings within the scope of a unified theory many kinds of complexes and other structures. One aspect of such objects is investigated in the theory of "regular complex polytopes" initiated by Shephard [1952]; for a complete account see Coxeter [1974]. Other directions and points of view are considered in Grünbaum [1975].

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